



## Some properties of rapidly varying functions<sup>☆</sup>



Nebojša Elez<sup>a</sup>, Dragan Djurčić<sup>b,\*</sup>

<sup>a</sup> University of East Sarajevo, Faculty of Philosophy, Alekse Šantića 1, 71420 Pale, Bosnia and Herzegovina

<sup>b</sup> University in Kragujevac, Technical Faculty, Svetog Save 65, 32000 Čačak, Serbia

### ARTICLE INFO

#### Article history:

Received 19 September 2012  
Available online 13 December 2012  
Submitted by Kathy Driver

#### Keywords:

Rapid variability  
Lower generalized inverse  
Upper generalized inverse

### ABSTRACT

In this paper some characterizations of the class of rapidly varying functions using the notions of the lower and upper generalized inverses will be proved. The important properties of this class that are related to two classical integral transformations will be proved, also.

© 2012 Elsevier Inc. All rights reserved.

### 1. Introduction and results

A measurable function  $f: [a, +\infty) \mapsto (0, +\infty)$  ( $a > 0$ ) is called slowly varying in the sense of Karamata (see e.g. [10]) if it satisfies the following condition:

$$\lim_{x \rightarrow +\infty} \frac{f(\lambda x)}{f(x)} = 1 \quad (1)$$

for every  $\lambda > 0$ . The class of all these functions (denoted by  $SV$ ) is the main object in Karamata's theory (see e.g. [1]).

A measurable function  $f: [a, +\infty) \mapsto (0, +\infty)$  ( $a > 0$ ) is called rapidly varying in the sense of de Haan with the index of variability  $+\infty$  (see e.g. [3]) if it satisfies the following condition:

$$\lim_{x \rightarrow +\infty} \frac{f(\lambda x)}{f(x)} = +\infty \quad (2)$$

for every  $\lambda > 1$ .

This functional class is denoted by  $R_\infty$ . The theory of rapid variability (with its generalizations) is an important part of asymptotical analysis and game theory (see e.g. [1,4,7,5,9,11]).

**Remark 1.1.** In this paper we will consider an elements from classes  $SV$  and  $R_\infty$  defined in  $(0, +\infty)$ , without loss of generality.

Let  $F^{(\infty)}$  be the set of all functions  $f: (0, +\infty) \mapsto (0, +\infty)$  which are bounded in  $(0, \alpha)$  for every  $\alpha \in (0, +\infty)$ , and for which  $\limsup_{x \rightarrow +\infty} f(x) = +\infty$  (see [2]).

<sup>☆</sup> This paper was supported by the Ministry of Education, Science and Technological Development of the Republic of Serbia, Grant No. 174032.

\* Corresponding author.

E-mail addresses: [jasnaelez@gmail.com](mailto:jasnaelez@gmail.com) (N. Elez), [dragandj@tfc.kg.ac.rs](mailto:dragandj@tfc.kg.ac.rs) (D. Djurčić).

For any  $f \in F^{(\infty)}$ , the positive, nondecreasing and unbounded function

$$f^{\leftarrow}(y) = \inf\{x > 0 \mid f(x) > y\} \tag{3}$$

defined in  $(b, +\infty)$  for every  $y > b = \inf\{f(t) \mid t \in (0, +\infty)\} \geq 0$  is its generalized inverse (see e.g. [1]). It is a very important object in asymptotic analysis (see e.g. [7,8,6,5]) and it can be used to characterize relationships between functional classes  $SV$  and  $R_\infty$  (see e.g. [1,7]).

Let  $F^\infty = \{f \in F^{(\infty)} \mid \liminf_{t \rightarrow +\infty} f(t) = +\infty\}$ . Now, for any  $f \in F^\infty$ , we will consider the following two positive and nondecreasing functions:

$$1^\circ f^{\leftarrow l}(y) = \inf\{x > 0 \mid f(x) \geq y\}, \text{ and}$$

$$2^\circ f^{\leftarrow u}(y) = \sup\{x > 0 \mid f(x) \leq y\},$$

for every  $y > b$ . These functions are important generalizations of the generalized inverse for elements from  $F^\infty$  (see e.g. [2]). Actually,  $f^{\leftarrow l}(y) \leq f^{\leftarrow}(y) \leq f^{\leftarrow u}(y)$  is satisfied for any  $f \in F^\infty$ , and every  $y \in (b, +\infty)$ . Furthermore, if  $f$  is a continuous and strictly increasing function, the previous inequalities become equalities and the observed value is equal to  $f^{-1}(y)$ , where  $f^{-1}$  is the inverse of a function  $f$ . It can be proved that  $f^{\leftarrow l}(f(x)) \leq x \leq f^{\leftarrow u}(f(x))$  is satisfied for any  $f \in F^\infty$  and every  $x > 0$ . Also, it can be proved that the following two assertions:

1.  $x < f^{\leftarrow l}(y)$  if and only if  $\bar{f}(x) < y$  and
2.  $f^{\leftarrow u}(y) < x$  if and only if  $\underline{f}(x) > y$

are satisfied for  $x > 0$  and  $y > b$ , where

$$3^\circ \bar{f}(x) = \sup\{f(t) \mid t \leq x\} \text{ and}$$

$$4^\circ \underline{f}(x) = \inf\{f(t) \mid t \geq x\},$$

for every  $x > 0$ . More about functions  $\bar{f}$  and  $\underline{f}$ , for  $f \in F^\infty$ , can be found in [1]. Here we note that these functions are positive and nondecreasing and it holds that  $\underline{f}(x) \leq f(x) \leq \bar{f}(x)$  for every  $x > 0$ . Thus, it can be concluded that  $f^{\leftarrow l}(y) = \sup\{x > 0 \mid \bar{f}(x) < y\}$  and  $f^{\leftarrow u}(y) = \inf\{x > 0 \mid y < \underline{f}(x)\}$ , for any  $f \in F^\infty$  and every  $y > b$ .

In the following theorem we give multiple characterizations of functions from the class  $R_\infty$  which belong to the class  $F^\infty$ .

**Theorem 1.1.** *Let  $f \in F^\infty$  be a measurable function. The following assertions are mutually equivalent:*

- (a) a function  $f$  belongs to  $R_\infty$ ;
- (b)  $\lim_{x \rightarrow +\infty} \inf_{\lambda \geq L} \frac{f(x)}{f(\frac{x}{\lambda})} = +\infty$  for every  $L > 1$ ;
- (c)  $\lim_{x \rightarrow +\infty} \frac{f(\lambda x)}{\bar{f}(x)} = +\infty$  for every  $\lambda > 1$ ;
- (d)  $\lim_{y \rightarrow +\infty} \frac{f^{\leftarrow u}(\lambda y)}{f^{\leftarrow l}(y)} = 1$  for every  $\lambda > 1$ ;
- (e) let  $g: (b, +\infty) \mapsto (0, +\infty)$  be a measurable function such that  $f^{\leftarrow l}(y) \leq g(y) \leq f^{\leftarrow u}(y)$  for every  $y > b$ ; then the function  $g \in SV$ ;
- (f) a function  $f^{\leftarrow l}$  belongs to  $SV$  and  $f^{\leftarrow l}(y) \sim f^{\leftarrow u}(y)$  for  $y \rightarrow +\infty$  (where  $\sim$  is the strong asymptotic equivalence relation (see e.g. [1])).

In the following theorem we give some properties for elements from the class  $SV \cap F^\infty$  (which are analogous to properties given in the previous theorem (assertions (d) and (e)) for elements from the class  $R_\infty \cap F^\infty$ ).

**Theorem 1.2.** *Let  $g \in SV \cap F^\infty$ . Then the following assertions hold:*

- (a)  $\lim_{y \rightarrow +\infty} \frac{g^{\leftarrow l}(\lambda y)}{g^{\leftarrow u}(y)} = +\infty$  for every  $\lambda > 1$ ;
- (b) every measurable function  $f: [b, +\infty) \mapsto (0, +\infty)$  such that  $g^{\leftarrow l}(y) \leq f(y) \leq g^{\leftarrow u}(y)$  for every  $y > b$  belongs to the class  $R_\infty$ .

Now, we consider an interesting equivalence relation for the class  $R_\infty$ . Let  $f$  and  $g$  be positive functions in  $(0, +\infty)$ . For these functions we say that they are mutually rapidly equivalent (denoted by  $f(x) \overset{r}{\sim} g(x)$  for  $x \rightarrow +\infty$ ) if the condition

$$\lim_{x \rightarrow +\infty} \frac{f(\lambda x)}{g(x)} = \lim_{x \rightarrow +\infty} \frac{g(\lambda x)}{f(x)} = +\infty, \tag{4}$$

is satisfied for every  $\lambda > 1$ . More about relation (4) can be found in [1,8,6].

**Theorem 1.3.** *Let  $f$  and  $g$  be a positive functions in  $(0, +\infty)$ . Then the following assertions hold:*

- (a) if  $f$  and  $g$  are measurable functions such that  $f(x) \overset{r}{\sim} g(x)$  for  $x \rightarrow +\infty$ , then  $f, g$  belong to  $R_\infty$ ;
- (b) the relation  $\overset{r}{\sim}$  is an equivalence relation in the class  $R_\infty$ ;

- (c) let  $f \in R_\infty$  and let  $f(x) \asymp g(x)$  for  $x \rightarrow +\infty$  (where  $\asymp$  is the weak asymptotic equivalence relation (see e.g. [1])); then  $f(x) \overset{r}{\sim} g(x)$  for  $x \rightarrow +\infty$ ;
- (d) let  $f$  be a measurable function; a function  $f$  belongs to  $R_\infty$  if and only if  $f(x) \overset{r}{\sim} \bar{f}(x)$  for  $x \rightarrow +\infty$ ;
- (e) let  $f$  belong to  $R_\infty$  and let  $f(x) \leq g(x) \leq \bar{f}(x)$  for  $x \geq x_0 > 0$ ; if  $g$  is a measurable function then  $g$  belongs to  $R_\infty$ .

Now, we will consider two integral transformations in the class  $R_\infty$ .

For a measurable function  $f: (0, +\infty) \mapsto (0, +\infty)$  such that  $f(x)$  is a bounded function in  $(0, x)$  for every  $x > 0$ , we define the transformation

$$f(x) \underset{\sim}{=} \frac{1}{x} \int_0^x f(t) dt = \int_0^1 f(xu) du \tag{5}$$

for  $x > 0$ . On the other hand, for a measurable function  $f: (0, +\infty) \mapsto (0, +\infty)$  such that  $\frac{1}{f(x)}$  is a bounded function in  $(0, x)$  for every  $x > 0$ , we define the transformation

$$\tilde{f}(x) = \frac{1}{x \int_x^{+\infty} \frac{dt}{t^2 f(t)}} = \frac{1}{\int_0^1 \frac{du}{f(\frac{x}{u})}} \tag{6}$$

for  $x > 0$ . More about transformations (5) and (6) can be found in [1].

**Theorem 1.4.** Let  $f \in R_\infty$  be a bounded function in  $(0, x)$  for every  $x > 0$ . Also, let  $\frac{1}{f(x)}$  be a bounded function for every  $x > 0$ . Then the following assertions hold:

- (a)  $f(x) \overset{r}{\sim} f(x) \overset{r}{\sim} \tilde{f}(x)$  for  $x \rightarrow +\infty$ ;
- (b) functions  $f, \tilde{f} \in R_\infty$ ;
- (c)  $f(x) \sim \int_{1-\varepsilon}^1 f(xu) du$  for  $x \rightarrow +\infty$  and every  $\varepsilon \in (0, 1]$ , and  $\frac{1}{f(x)} \sim \int_{1-\varepsilon}^1 \frac{du}{f(\frac{x}{u})}$  for  $x \rightarrow +\infty$  and every  $\varepsilon \in (0, 1]$  (where  $\sim$  is the strong asymptotic equivalence relation (see e.g. [1]));
- (d)  $f(x) = o(\tilde{f}(x))$  for  $x \rightarrow +\infty$ , and  $\tilde{f}(x) = o(f(x))$  for  $x \rightarrow +\infty$  (where  $o$  is the Landau symbol (see e.g. [1])).

**Corollary 1.1.** (a) Let  $f \in R_\infty$  be a function with the same properties as in Theorem 1.4. Then it holds that the function  $\int_0^x f(t) dt \in R_\infty$  for  $x > 0$ , and the function  $\frac{1}{\int_x^{+\infty} \frac{dt}{f(t)}} \in R_\infty$  for  $x > 0$ . Also, it holds that  $\int_0^x f(t) dt \sim \int_{\frac{x}{\lambda}}^x f(t) dt$  for  $x \rightarrow +\infty$  and every  $\lambda > 1$ , and  $\int_x^{+\infty} \frac{dt}{f(t)} \sim \int_x^{\lambda x} \frac{dt}{f(t)}$  for  $x \rightarrow +\infty$  and every  $\lambda > 1$ .

(b) Let  $f \in R_\infty$  be a nondecreasing function. Then  $f(x) = o(\tilde{f}(x))$  for  $x \rightarrow +\infty$ , and  $f(x) = o(\tilde{f}(x))$  for  $x \rightarrow +\infty$ .

**Theorem 1.5.** Let  $f: (0, +\infty) \mapsto (0, +\infty)$  be a nondecreasing function.

- (a) If  $f(x) = o(f(x))$  for  $x \rightarrow +\infty$ , then  $f \in R_\infty$ .
- (b) If  $f(x) = o(\tilde{f}(x))$  for  $x \rightarrow +\infty$ , then  $f \in R_\infty$ .

The set of all measurable functions  $f: (0, +\infty) \mapsto (0, +\infty)$  whose lower Matuszewska index is equal to  $+\infty$  is denoted by  $MR_\infty$  (see e.g. [1]). It is well-known that  $MR_\infty \subsetneq R_\infty$ . Hence, if we observe a function from the set  $MR_\infty$  in Theorem 1.4(d), we will get stronger conclusions than we got for a function from the set  $R_\infty$ . We discuss this in the following theorem.

**Theorem 1.6.** Let  $f \in MR_\infty$  be a function with the same properties as in the assumptions of Theorem 1.4. Then  $f(x) = o(\tilde{f}(x))$  for  $x \rightarrow +\infty$ , and  $f(x) = o(\tilde{f}(x))$  for  $x \rightarrow +\infty$ .

**Example 1.1.** (1) Let

$$f(x) = \begin{cases} e^x, & \text{for } x \in (0, +\infty) \setminus \mathbb{N}; \\ xe^x, & \text{for } x \in \mathbb{N}. \end{cases}$$

Then  $f \in R_\infty \setminus MR_\infty$  and  $f(x) \neq o(f(x))$  for  $x \rightarrow +\infty$ .

(2) Let

$$f(x) = \begin{cases} \frac{e^x}{x^2}, & \text{for } x \in (0, +\infty) \setminus \mathbb{N}; \\ \frac{e^x}{x}, & \text{for } x \in \mathbb{N}. \end{cases}$$

Then  $f \in R_\infty \setminus MR_\infty$  and  $f(x) \neq o(\tilde{f}(x))$  for  $x \rightarrow +\infty$ .

2. Proofs

**Proof of Theorem 1.1.** ((a) ⇒ (b)) Let  $M > 1$ . On the basis of the theorem of uniform convergence for the rapidly varying functions (see e.g. [1]), for every  $L > 1$  there exists  $x_0 > 0$  such that for every  $x \geq x_0$  and every  $\lambda > L$  it holds that  $\frac{f(\lambda x)}{f(x)} > M$ . Now, there is an  $x_1 > 0$  such that  $f(x) \geq M \cdot \sup\{f(t) \mid t \in (0, x_0]\}$  is satisfied for every  $x \geq x_1$ . Then the following assertions hold:

1. if  $\frac{x}{\lambda} \leq x_0$ , then  $\frac{f(x)}{f(\frac{x}{\lambda})} \geq \frac{f(x)}{\sup\{f(t) \mid t \in (0, x_0]\}} \geq M$ ;
2. if  $\frac{x}{\lambda} \geq x_0$ , then  $\frac{f(x)}{f(\frac{x}{\lambda})} = \frac{f(\lambda \frac{x}{\lambda})}{f(\frac{x}{\lambda})} \geq M$ ;

for every  $\lambda > L$  and  $x > x_1$ . In any case, it holds that  $\inf\left\{\frac{f(x)}{f(\frac{x}{\lambda})} \mid \lambda \geq L, x \geq x_1\right\} \geq M$ , and we obtain  $\lim_{x \rightarrow +\infty} \inf_{\lambda \geq L} \frac{f(x)}{f(\frac{x}{\lambda})} = +\infty$ .

((b) ⇒ (c)) Let  $\lambda > 1$ . Then for  $x > 0$  it holds that

$$\begin{aligned} \frac{f(\lambda x)}{\bar{f}(x)} &= \frac{\inf\{f(t) \mid t \geq \lambda x\}}{\sup\{f(s) \mid s \in (0, x]\}} \\ &= \inf\left\{\frac{f(t)}{f(s)} \mid s \in (0, x], t \geq \lambda x\right\} \\ &\geq \inf\left\{\frac{f(t)}{f\left(\frac{t}{\mu}\right)} \mid \mu \geq \lambda, t \geq \lambda x\right\}. \end{aligned}$$

Hence, we obtain  $\liminf_{x \rightarrow +\infty} \frac{f(\lambda x)}{\bar{f}(x)} \geq \lim_{t \rightarrow +\infty} \inf_{\mu \geq \lambda} \frac{f(t)}{f\left(\frac{t}{\mu}\right)} = +\infty$ .

((c) ⇒ (a)) Let  $\lambda > 1$  and  $x > 0$ . Then from  $\frac{f(\lambda x)}{f(x)} \geq \frac{f(\lambda x)}{\bar{f}(x)}$ , we obtain  $\lim_{x \rightarrow +\infty} \frac{f(\lambda x)}{f(x)} = +\infty$ , i.e.  $f \in R_\infty$ .

((a) ⇒ (d)) Let  $\lambda > 1$ . Then it holds that

$$\begin{aligned} \frac{f^{\leftarrow u}(\lambda x)}{f^{\leftarrow l}(x)} &= \frac{\sup\{s \mid f(s) \leq \lambda x\}}{\inf\{t \mid f(t) \geq x\}} \\ &= \sup\left\{\frac{s}{t} \mid f(s) \leq \lambda x, f(t) \geq x\right\} \\ &\leq \sup\left\{\mu > 0 \mid \frac{f(\mu t)}{f(t)} \leq \lambda, f(t) \geq x\right\}. \end{aligned}$$

From results obtained in [9] it follows that

$$\limsup_{x \rightarrow +\infty} \frac{f^{\leftarrow u}(\lambda x)}{f^{\leftarrow l}(x)} \leq \limsup_{t \rightarrow +\infty} \sup\left\{\mu > 0 \mid \frac{f(\mu t)}{f(t)} \leq \lambda\right\} \leq 1.$$

On the other hand, it is obvious that

$$\frac{f^{\leftarrow u}(\lambda x)}{f^{\leftarrow l}(x)} \geq \frac{f^{\leftarrow u}(x)}{f^{\leftarrow l}(x)} \geq 1,$$

is satisfied for the same  $x$  and  $\lambda$ . Hence,  $\lim_{x \rightarrow +\infty} \frac{f^{\leftarrow u}(\lambda x)}{f^{\leftarrow l}(x)} = 1$  for  $\lambda > 1$ .

((d) ⇒ (a)) Let assertion (d) hold. We will assume that  $f \notin R_\infty$ . Then from results obtained in [9] there is a sequence of real numbers  $(x_n)$  such that  $\lim_{n \rightarrow +\infty} x_n = +\infty$ , and there is a sequence of positive real numbers  $(\lambda_n)$  such that  $\liminf_{n \rightarrow +\infty} \lambda_n > \lambda > 1$ , and for those two sequences it holds that  $\limsup_{n \rightarrow +\infty} \frac{f(\lambda_n x_n)}{f(x_n)} < \mu < +\infty$  for some  $\lambda > 1$  and  $\mu > 1$ . Furthermore, inequalities  $f^{\leftarrow l}(f(x_n)) \leq x_n$  and  $f^{\leftarrow u}(\mu f(x_n)) = \sup\{x > 0 \mid f(x) \leq \mu \cdot f(x_n)\} \geq \lambda_n \cdot x_n > \lambda \cdot x_n$  are satisfied for sufficiently large  $n$  (because for sufficiently large  $n$  it holds that  $f(\lambda_n x_n) \leq \mu \cdot f(x_n)$ ) which implies that

$$\limsup_{y \rightarrow +\infty} \frac{f^{\leftarrow u}(\mu y)}{f^{\leftarrow l}(y)} \geq \limsup_{n \rightarrow +\infty} \frac{f^{\leftarrow u}(\mu f(x_n))}{f^{\leftarrow l}(f(x_n))} \geq \limsup_{n \rightarrow +\infty} \frac{\lambda \cdot x_n}{x_n} > 1.$$

The last inequality is a contradiction to assertion (d). Hence,  $f \in R_\infty$ .

((d) ⇒ (e)) From  $f^{\leftarrow l}(y) \leq g(y) \leq f^{\leftarrow u}(y)$  for every  $y \in (b, +\infty)$ , we have that

$$F(\lambda, y) = \frac{f^{\leftarrow u}(\lambda y)}{f^{\leftarrow l}(y)} \geq \frac{g(\lambda y)}{g(y)} \geq \frac{f^{\leftarrow l}(\lambda y)}{f^{\leftarrow u}(y)} \geq \frac{f^{\leftarrow l}(y)}{f^{\leftarrow u}(\lambda y)} = \frac{1}{F(\lambda, y)}$$

is satisfied for the same  $y$  and every  $\lambda > 1$ . Hence,  $\lim_{y \rightarrow +\infty} \frac{g(\lambda y)}{g(y)} = 1$  for the same  $\lambda$ , and we obtain that  $g \in SV$ , because  $g$  is a measurable function.

((e) ⇒ (f)) Trivially, on the basis of assertion (e) it holds that  $f^{\leftarrow l} \in SV$  (similarly,  $f^{\leftarrow u} \in SV$ ). Now, we will assume that

$$g(y) = \begin{cases} f^{\leftarrow l}(y), & \text{for } y \geq b, y \in \mathbb{Q}, \\ f^{\leftarrow u}(y), & \text{for } y \geq b, y \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Then  $g \in SV$ , and it holds that

$$1 \leq \lim_{y \rightarrow +\infty} \frac{f^{\leftarrow u}(\lambda y)}{f^{\leftarrow l}(y)} \leq \lim_{y \rightarrow +\infty} \sup_{\frac{1}{\lambda} \leq \mu \leq \lambda} \frac{g(\mu y)}{g(y)} = 1$$

for  $\lambda > 1$ . Hence, we obtain  $f^{\leftarrow l}(y) \sim f^{\leftarrow u}(y)$  for  $y \rightarrow +\infty$ .

((f) ⇒ (d)) Let  $\lambda > 1$ . On the basis of assertion (f) it follows that

$$\lim_{y \rightarrow +\infty} \frac{f^{\leftarrow u}(\lambda y)}{f^{\leftarrow l}(y)} = \lim_{y \rightarrow +\infty} \frac{f^{\leftarrow u}(\lambda y)}{f^{\leftarrow l}(\lambda y)} \cdot \lim_{y \rightarrow +\infty} \frac{f^{\leftarrow l}(\lambda y)}{f^{\leftarrow l}(y)} = 1.$$

Finally, assertion (d) holds. □

**Proof of Theorem 1.2.** (a) Let  $\lambda > 1$ . Then it holds that

$$\begin{aligned} \frac{g^{\leftarrow l}(\lambda x)}{g^{\leftarrow u}(x)} &= \frac{\inf\{s \mid g(s) \geq \lambda x\}}{\sup\{t \mid g(t) \leq x\}} \\ &= \inf\left\{\frac{s}{t} \mid g(t) \leq x, g(s) \geq \lambda x\right\} \\ &\geq \inf\left\{\mu > 0 \mid \frac{g(\mu t)}{g(t)} \geq \lambda, g(t) \leq x\right\} \end{aligned}$$

for  $x \geq b$ . Since  $g \in SV$ , we obtain that

$$\liminf_{x \rightarrow +\infty} \frac{g^{\leftarrow l}(\lambda x)}{g^{\leftarrow u}(x)} \geq \liminf_{t \rightarrow +\infty} \inf\left\{\mu > 0 \mid \frac{g(\mu t)}{g(t)} \geq \lambda\right\} = +\infty.$$

This completes the proof of assertion (a).

(b) Let  $\lambda > 1$ . Since  $f$  is a measurable function we have that

$$\lim_{x \rightarrow +\infty} \frac{f(\lambda x)}{f(x)} \geq \lim_{x \rightarrow +\infty} \frac{g^{\leftarrow l}(\lambda x)}{g^{\leftarrow u}(x)} = +\infty$$

is satisfied. Hence,  $f \in R_\infty$ . □

**Proof of Theorem 1.3.** (a) Let  $f, g$  be a measurable functions such that  $f(x) \overset{r}{\sim} g(x)$  for  $x \rightarrow +\infty$ . Then

$$\lim_{x \rightarrow +\infty} \frac{f(\lambda x)}{f(x)} = \lim_{x \rightarrow +\infty} \frac{f(\lambda x)}{g(\sqrt{\lambda} x)} \cdot \lim_{x \rightarrow +\infty} \frac{g(\sqrt{\lambda} x)}{f(x)} = +\infty,$$

is satisfied for  $\lambda > 1$ . Therefore,  $f \in R_\infty$  and similarly  $g \in R_\infty$ .

(b) We will prove only that relation  $\overset{r}{\sim}$  is the transitive relation (reflexivity and symmetry of this relation are obvious). Let  $f(x) \overset{r}{\sim} g(x)$  for  $x \rightarrow +\infty$ , and  $g(x) \overset{r}{\sim} h(x)$  for  $x \rightarrow +\infty$ . Therefore, we have that the equalities

$$\lim_{x \rightarrow +\infty} \frac{f(\lambda x)}{h(x)} = \lim_{x \rightarrow +\infty} \frac{f(\lambda x)}{g(\sqrt{\lambda} x)} \cdot \lim_{x \rightarrow +\infty} \frac{g(\sqrt{\lambda} x)}{h(x)} = +\infty$$

and

$$\lim_{x \rightarrow +\infty} \frac{h(\lambda x)}{f(x)} = \lim_{x \rightarrow +\infty} \frac{h(\lambda x)}{g(\sqrt{\lambda} x)} \cdot \lim_{x \rightarrow +\infty} \frac{g(\sqrt{\lambda} x)}{f(x)} = +\infty$$

are satisfied, for  $\lambda > 1$ . Using assertion (a) we obtain that this assertion holds.

(c) We have that

$$\liminf_{x \rightarrow +\infty} \frac{f(\lambda x)}{g(x)} \geq \liminf_{x \rightarrow +\infty} \frac{f(\lambda x)}{f(x)} \cdot \liminf_{x \rightarrow +\infty} \frac{f(x)}{g(x)} = +\infty$$

for  $\lambda > 1$ . Since  $f(x) \asymp g(x)$  for  $x \rightarrow +\infty$ , it follows that

$$\liminf_{x \rightarrow +\infty} \frac{g(\lambda x)}{f(x)} \geq \liminf_{x \rightarrow +\infty} \frac{g(\lambda x)}{f(\lambda x)} \cdot \liminf_{x \rightarrow +\infty} \frac{f(\lambda x)}{f(x)} = +\infty$$

for  $\lambda > 1$ . Hence,  $f(x) \stackrel{r}{\sim} g(x)$  for  $x \rightarrow +\infty$ .

(d) ( $\Rightarrow$ ) Let  $f \in R_\infty$ . Then, from [Theorem 1.1](#) it holds that  $\lim_{x \rightarrow +\infty} \frac{f(\lambda x)}{\bar{f}(x)} = +\infty$  for  $\lambda > 1$ . From previous computations, it follows that  $\lim_{x \rightarrow +\infty} \frac{\bar{f}(\lambda x)}{\underline{f}(x)} = +\infty$ , i.e.  $\bar{f}(x) \stackrel{r}{\sim} \underline{f}(x)$ , for  $x \rightarrow +\infty$ .

( $\Leftarrow$ ) Let  $\bar{f}(x) \stackrel{r}{\sim} \underline{f}(x)$  for  $x \rightarrow +\infty$ . Then, it follows that  $\lim_{x \rightarrow +\infty} \frac{f(\lambda x)}{\bar{f}(x)} = +\infty$  for  $\lambda > 1$ , and applying [Theorem 1.1](#) we obtain  $f \in R_\infty$ .

(e) Let  $f \in R_\infty$ . Then, it holds that

$$\liminf_{x \rightarrow +\infty} \frac{g(\lambda x)}{g(x)} \geq \liminf_{x \rightarrow +\infty} \frac{f(\lambda x)}{\bar{f}(x)} = +\infty$$

for  $\lambda > 1$ . Applying [Theorem 1.1](#) and previous computations, we obtain  $g \in R_\infty$ .  $\square$

**Proof of Theorem 1.4.** (a) Let  $\lambda > 1$ . It is sufficient to prove that

$$\lim_{x \rightarrow +\infty} \frac{\tilde{f}(x)}{f(\lambda x)} = \lim_{x \rightarrow +\infty} \frac{f(x)}{\tilde{f}(\lambda x)} = \lim_{x \rightarrow +\infty} \frac{\tilde{f}(x)}{\tilde{f}(\lambda x)} = 0.$$

We have that

$$\frac{\tilde{f}(x)}{f(\lambda x)} = \int_0^1 \frac{f(ux)}{f(\lambda x)} du \rightarrow 0$$

is satisfied for  $x \rightarrow +\infty$ , because it holds that  $\lim_{x \rightarrow +\infty} \frac{f(ux)}{f(\lambda x)} = 0$  for  $\lambda > 1$  and  $u \in (0, 1]$ , and  $\sup_{0 < u \leq 1} \left| \frac{f(ux)}{f(\lambda x)} \right| \leq 1$  for sufficiently large  $x$  is satisfied (based on [Theorem 1.1\(b\)](#)). Now, we can apply Lebesgue's theorem of dominant convergence. According to this theorem, it holds that

$$\frac{f(x)}{\tilde{f}(\lambda x)} = \int_0^1 \frac{f(x)}{f\left(\frac{\lambda x}{u}\right)} du \rightarrow 0 \tag{7}$$

for  $x \rightarrow +\infty$ . Actually, condition (7) is satisfied because it holds that  $\lim_{x \rightarrow +\infty} \frac{f(x)}{f\left(\frac{\lambda x}{u}\right)} = 0$  for  $\lambda > 1$  and  $u \in (0, 1]$ , and

$\sup_{0 < u \leq 1} \left| \frac{f(x)}{f\left(\frac{\lambda x}{u}\right)} \right| \leq 1$  for sufficiently large  $x$  is satisfied (on the basis of [Theorem 1.1\(b\)](#)).

Now, let  $\lambda > 1$  and  $M \in (0, +\infty)$ . We have that  $\frac{\lambda u}{u} > \sqrt{\lambda}$  is satisfied for  $u \in (\lambda^{-\frac{1}{4}}, 1)$ . Hence, it holds that  $\frac{f(\lambda ux)}{f\left(\frac{\lambda x}{u}\right)} > \frac{M}{(1-\lambda^{-\frac{1}{4}})^2}$  for sufficiently large  $x$ . Applying the Cauchy–Schwarz inequality we obtain that it holds that

$$\begin{aligned} \frac{\tilde{f}(\lambda x)}{\tilde{f}(x)} &= \int_0^1 f(\lambda ux) du \cdot \int_0^1 \frac{du}{f\left(\frac{\lambda x}{u}\right)} \geq \left( \int_0^1 \sqrt{\frac{f(\lambda ux)}{f\left(\frac{\lambda x}{u}\right)}} du \right)^2 \\ &\geq \left( \int_{\lambda^{-\frac{1}{4}}}^1 \sqrt{\frac{f(\lambda ux)}{f\left(\frac{\lambda x}{u}\right)}} du \right)^2 \geq M \end{aligned}$$

for sufficiently large  $x$ . Hence,  $\lim_{x \rightarrow +\infty} \frac{\tilde{f}(x)}{\tilde{f}(\lambda x)} = 0$ .

(b) Let  $\lambda > 1$ . Then from assertion (a), it holds that

$$\lim_{x \rightarrow +\infty} \frac{\tilde{f}(\lambda x)}{\tilde{f}(x)} = \lim_{x \rightarrow +\infty} \left( \frac{\tilde{f}(\lambda x)}{\tilde{f}(\lambda^{\frac{2}{3}} x)} \cdot \frac{\tilde{f}(\lambda^{\frac{2}{3}} x)}{f(\lambda^{\frac{1}{3}} x)} \cdot \frac{f(\lambda^{\frac{1}{3}} x)}{\tilde{f}(x)} \right) = +\infty.$$

Furthermore,  $f$  is a measurable and positive function in  $(0, +\infty)$  which implies that  $f \in R_\infty$ . Analogously, it can be shown that  $\tilde{f} \in R_\infty$ . This assertion can be proved by combining the results obtained in Theorems 1.4(a) and 1.3(a).

(c) Let  $\varepsilon > 0$  and  $\lambda = \frac{1}{1-\varepsilon} > 1$ . Since  $f \in R_\infty$ , it follows that  $\lim_{x \rightarrow +\infty} \frac{f(\frac{x}{\lambda})}{\tilde{f}(x)} = 0$ . Furthermore, it holds that

$$\int_0^{1-\varepsilon} f(ux) du = \lambda^{-1} \int_0^1 f\left(\frac{t}{\lambda}x\right) dt = o\left(\int_0^1 f(tx) dt\right)$$

for  $x \rightarrow +\infty$ . Hence,  $\int_{1-\varepsilon}^1 f(ux) du \sim \tilde{f}(x)$  for  $x \rightarrow +\infty$ . Analogously, it can be shown that  $\frac{1}{\tilde{f}(x)} \sim \int_{1-\varepsilon}^1 \frac{du}{f(\frac{x}{u})}$  for  $x \rightarrow +\infty$ .

(d) We have that  $\lim_{x \rightarrow +\infty} \frac{f(ux)}{\tilde{f}(x)} = 0$  is satisfied for  $u \in (0, 1)$  (because  $f(x) \sim \tilde{f}(x)$  for  $x \rightarrow +\infty$ ). Also,  $\sup_{0 < u \leq 1} \left| \frac{f(ux)}{\tilde{f}(x)} \right| = 1$  is satisfied for  $x > 0$ . Then, it follows that  $\lim_{x \rightarrow +\infty} \frac{f(x)}{\tilde{f}(x)} = \lim_{x \rightarrow +\infty} \int_0^1 \frac{f(ux)}{\tilde{f}(x)} du = 0$  according to Lebesgue's theorem of dominant convergence. According to this theorem, we have that  $\lim_{x \rightarrow +\infty} \frac{f(x)}{\tilde{f}(x)} = \lim_{x \rightarrow +\infty} \int_0^1 \frac{f(x)}{\tilde{f}(\frac{x}{u})} du = 0$  is satisfied, because  $\lim_{x \rightarrow +\infty} \frac{f(x)}{\tilde{f}(\frac{x}{u})} = 0$  is satisfied for  $u \in (0, 1)$  ( $f(x) \sim \tilde{f}(x)$  for  $x \rightarrow +\infty$ ), and also  $\sup_{0 < u \leq 1} \left| \frac{f(x)}{\tilde{f}(\frac{x}{u})} \right| = 1$  is satisfied for  $x > 0$ .  $\square$

**Proof of Theorem 1.5.** (a) Let us assume that  $f(x) = o(\tilde{f}(x))$  for  $x \rightarrow +\infty$  and  $f \notin R_\infty$ . Then, there is a sequence  $(x_n)$  (of positive numbers) such that  $x_n \rightarrow +\infty$  for  $n \rightarrow +\infty$ , and there is a  $\lambda > 1$  such that  $\frac{f(\lambda x_n)}{f(x_n)} < M < +\infty$  for some  $M \in (0, +\infty)$  and every  $n \in \mathbb{N}$ . Also, it holds that

$$\int_{x_n}^{\lambda x_n} f(t) dt \leq \int_0^{\lambda x_n} f(t) dt = o(\lambda \cdot x_n \cdot f(\lambda x_n))$$

for  $n \rightarrow +\infty$ . From the foregoing, it follows that

$$\int_{x_n}^{\lambda x_n} f(t) dt = o(x_n \cdot f(x_n))$$

for  $n \rightarrow +\infty$ , and from  $\int_{x_n}^{\lambda x_n} f(t) dt \geq (\lambda - 1) \cdot x_n \cdot f(x_n)$  for  $n \in \mathbb{N}$  we obtain that assumption  $f \notin R_\infty$  is wrong.

(b) Let us assume that  $f(x) = o(\tilde{f}(x))$  for  $x \rightarrow +\infty$  and  $f \notin R_\infty$ . Then there are a sequence  $(x_n)$  and a number  $\lambda$  with the same properties as in the proof of assertion (a). Also, the following two inequalities hold:

$$\int_{x_n}^{\lambda x_n} \frac{dt}{t^2 f(t)} \leq \int_{x_n}^{+\infty} \frac{dt}{t^2 f(t)} = o\left(\frac{1}{x_n \cdot f(x_n)}\right)$$

for  $n \rightarrow +\infty$ , and

$$\int_{x_n}^{\lambda x_n} \frac{dt}{t^2 f(t)} \geq \frac{(\lambda - 1)x_n}{(\lambda x_n)^2 f(\lambda x_n)} \geq \frac{\lambda - 1}{\lambda^2 M} \cdot \frac{1}{x_n f(x_n)}$$

for  $n \in \mathbb{N}$ . Finally, we obtain that assumption  $f \notin R_\infty$  is wrong.  $\square$

**Proof of Theorem 1.6.** We will prove this theorem using Lebesgue's theorem of dominant convergence. According to this theorem, we have that

$$\lim_{x \rightarrow +\infty} \frac{f(x)}{\tilde{f}(x)} = \lim_{x \rightarrow +\infty} \int_0^1 \frac{f(ux)}{f(x)} du = 0,$$

is satisfied, because it holds that  $\lim_{x \rightarrow +\infty} \frac{f(ux)}{f(x)} du = 0$  for  $u \in (0, 1)$ , and  $\sup_{0 < u \leq 1} \left| \frac{f(ux)}{f(x)} \right| < M$  is satisfied for some  $M \in (0, +\infty)$  and sufficiently large  $x$  (the last inequality holds because  $\beta(f) = +\infty$ , where  $\beta(f)$  is the lower Matuszewska index (see e.g. [1]), and therefore  $\alpha(f) = +\infty$ , where  $\alpha(f)$  is upper Matuszewska index).

Similarly, we have that

$$\lim_{x \rightarrow +\infty} \frac{f(x)}{\tilde{\tilde{f}}(x)} = \lim_{x \rightarrow +\infty} \int_0^1 \frac{f(x)}{f(\frac{x}{u})} du = 0$$

is satisfied, because it holds that  $\lim_{x \rightarrow +\infty} \frac{f(x)}{f(\frac{x}{u})} du = 0$  for  $u \in (0, 1)$ , and  $\inf_{\lambda > 1} \left| \frac{f(\lambda x)}{f(x)} \right| \geq \frac{1}{M}$  is satisfied for some  $M \in (0, +\infty)$  and sufficiently large  $x$  (this inequality holds because  $\beta(f) = +\infty$ ). Hence,  $\sup_{0 < u < 1} \left| \frac{f(x)}{f(\frac{x}{u})} \right| \leq M$ .  $\square$

## References

- [1] N.H. Bingham, C.M. Goldie, J.L. Teugels, *Regular Variation*, Cambridge Univ. Press, Cambridge, 1987.
- [2] V.V. Buldygin, O.I. Klesov, J.G. Steinebach, Some properties of asymptotic quasiinverse functions and their applications I, *Teor. Imovir. Mat. Stat.* 70 (2004) 9–25. English transl. in *Theory Probab. Math. Statist.* 70 (2005) 11–28.
- [3] L. de Haan, On Regular Variation and its Applications to the Weak Convergence of Sample Extremes, in: *Math. Centre Tracts*, vol. 32, CWI, Amsterdam, 1970.
- [4] D. Djurčić, D.R. Kočinac, M.R. Žižović, Some properties of rapidly varying sequences, *J. Math. Anal. Appl.* 327 (2) (2007) 1297–1306.
- [5] D. Djurčić, R. Nikolić, A. Torgašev, The weak and strong asymptotic equivalence relations and the generalized inverse, *Lith. Math. J.* 51 (4) (2011) 472–476.
- [6] D. Djurčić, R. Nikolić, A. Torgašev, The weak asymptotic equivalence and the generalized inverse, *Lith. Math. J.* 50 (1) (2010) 34–42.
- [7] D. Djurčić, A. Torgašev, Some asymptotic relations for the generalized inverse, *J. Math. Anal. Appl.* 335 (2) (2007) 1397–1402.
- [8] D. Djurčić, A. Torgašev, S. Ješić, The strong asymptotic equivalence and the generalized inverse, *Siber. Math. J.* 49 (4) (2008) 786–795.
- [9] N. Elez, D. Djurčić, The rapidly varying functions, *J. Math. Anal. Appl.* (submitted for publication).
- [10] J. Karamata, Sur un mode de croissance régulière des fonctions, *Mathematica (Cluj)* 4 (1930) 38–53.
- [11] S. Matucci, P. Reháč, Rapidly varying decreasing solutions of half-linear difference equations, *Math. Comput. Modelling* 49 (7–8) (2009) 1692–1699.