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Contractive Linear Operators and their Applications in F - Cone Metric Fixed Point Theory

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Abstract

In this paper we introduce the notion of a F - cone metric space. Furthermore we define and study contractive bounded linear operators on complete F - normed space. We also present some fixed point results on F - cone metric space, with operator's contractive condition which generalizes some results from [6] and [11].

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1 Introduction

There has been a number of generalizations of metric space. One of such generalization is the notion of a cone metric space (under the name of K-metric space) initiated by several Russian authors (for historical notes see [16]) in mid-20th. L.-G. Huang and X. Zhang [6] re-introduced such spaces, but went further, defining convergent and Cauchy sequences in the terms of interior points of the underlying cone. The concept of a cone metric space is more general than that of a metric space (see [6] - Example 1 and [13] - Examples 1.1 and 1.2). They have proved some fixed point theorems for this class of spaces. Further fixed point results for this class of spaces were obtained by D. Ilić and V. Rakočević [7], Sh. Rezapour and R. Hamlbarani [11], C. Di Bari and P. Vetro

[3], S. M. Veazpour and P. Raja [13], M. Arshad, A. Azam and P. Vetro [2], M. Abbas and B. Rhoades [1], Z. Kadelburg, S. Radenović and V. Rakočević [8], K. Włodarczyk, R. Plebaniak and M. Doliński [15]... Applications of such results in the theory of integral equations and topological theory of set-valued dynamic systems can be found in the papers [2], [13] and [15].

In this paper we introduce the notion of a F - cone metric space. Furthermore we define and study contractive bounded linear operators on complete F - normed space. We also present some fixed point results on F - cone metric space, with operator's contractive condition which generalizes some results from [6] and [11].

2 Preliminary Notes

Let X be a metrizable linear topological space. Then there exists (see [12],[14] or [9]) a metric d on X which is equivalent with the original metric on X such that function $|.\| : X \to [0, +\infty)$ defined by $|x\| = d(x, 0)$ has the following properties:

1) $|x\| = 0$ if and only if $x = 0$;
2) $|x\| = |-x\|$;
3) $|x + y\| \leq |x\| + |y\|$;
4) $0 < \alpha < \beta$ implies $|\alpha x\| < |\beta x\|$.

The mapping $|.\|$ is said to be a *paranorm*. *Pranormed* space is an ordered pair $(E, |.\|)$ where X is a metric linear space and $|.\|$ a paranorm on X. If (E, d) is complete metric space, where $d(x, y) = |x - y\|$ then $(E, |.\|)$ is F - space

Definition 2.1 *Let E be a linear topological space. A subset P of E is called a cone if:*

1) P is closed, nonempty and $P \neq \{0\}$;

2) $a, b \in \mathbf{R}$, $a, b > 0$, and $x, y \in P$ imply $ax + by \in P$;

3) $P \cap (-P) = \{0\}$.

Given a cone, $P \subseteq E$ we define partial ordering $\leq$ on E with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int}P$ (interior of P).

Definition 2.2 *Let $(E, |.\|)$ be a F space and $P \subseteq E$ be a cone. P is a normal cone if and only if there exists real number $K > 0$ such that $x \leq y$ implies*

$$|x\| \leq K|y\| \tag{1}$$

for each $x, y \in P$.

The least positive K satisfying (1) is called the normal constant of P. In [11] Sh. Rezapour and R. Hamlbarani proved that $K \geq 1$, when E is a Banach space. This proof for normal cone in F - normed space is the same.

Definition 2.3 *Let $(E, |.\|)$ be a F - space and $P \subseteq E$ be a cone. P is a solid cone if and only if* $\text{int}P \neq \emptyset$.

In the following we always suppose that $(E, |.\|)$ is a F - space, P is a solid cone in E such that $\leq$ is partial ordering on E with respect to P. By I we denote identity operator on E i.e. $I(x) = x$ for each $x \in E$.

Now we introduce the notion of *F - cone metric spaces*.

Definition 2.4 *Let X be a nonempty set. Suppose that a mapping $d : X \times X \to E$ satisfies:*

1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a F - cone metric on X and (X, d) is called a F - cone metric space.

Cone metric spaces in Huang - Zhang sense [6] are included in our definition because every Banach space is complete F - normed space.

Definition 2.5 *Let (X, d) be a F - cone metric space, $x \in X$ and $0 \ll c$. Then*

$$B(x, c) = \{y \in X : d(x, y) \ll c\}.$$

Definition 2.6 *Let (X, d) be a solid F - cone metric space, $x \in X$ and (x_n) a sequence in X. Then*

1) (x_n) converges to x if for every $c \in \text{int}P$ there exists a positive integer N such that for all $n \geq N$ $d(x_n, x) \ll c$, we denote this by $\lim x_n = x$ or $x_n \to x$;

2) (x_n) is Cauchy sequences if for every $c \in \text{int}P$ there exists a positive integer N such that for all $m, n \geq N$ $d(x_m, x_n) \ll c$;

3) (X, d) is a complete cone metric space if every Cauchy sequence is convergent.

In this paper we need the following Lemma.

Lemma 2.7 *Let $c \in \text{int}P$, $0 \ll c$ and $\{x_n\}$ be a sequence in P, such that $x_n \to 0$. Then there exists a positive integer n_0 such that $n > n_0$ implies $x_n \ll c$.*

Proof: Let V be a symmetric neighborhood of zero such that $c + V \subseteq \mathrm{int}P$. So there exists positive integer n_0 such that $n > n_0$ implies $x_n \in V = -V$. Hence $c - x_n \in c - V = c + V \subseteq \mathrm{int}P$ which implies $x_n \ll c$. $\diamondsuit$

The next two statements extend the recent results of C. Di Bari and P. Vetro [3] obtained for cone metric spaces.

Lemma 2.8 *Let (X, d) be a F - cone metric space $x \in X$ and $\{x_n\}$ be a sequence in X. If $\lim |d(x_n, x)\| = 0$ then $\lim x_n = x$.*

Proof: Let $0 \ll c$. Let V be a symmetric neighborhood of zero such that $c + V \subseteq \mathrm{int}P$. Then there exists positive integer n_0 such that $n > n_0$ implies $d(x, x_n) \in V = -V$. So $c - d(x, x_n) \in c - V = c + V \subseteq \mathrm{int}P$. Then for all $n > n_0$ we have $d(x_n, x) \ll c$ which implies that $\lim x_n = x$. $\diamondsuit$

Lemma 2.9 *Let (X, d) be a F - cone metric space and $\{x_n\}$ be a sequence in X. If $\lim x_n = y$ and $\lim x_n = z$ then $y = z$.*

Proof: Let $c \in \mathrm{int}P$. Then there exists positive integer n_0 such that $n > n_0$ implies $d(x_n, x) \ll \frac{c}{2}$ and $d(x_n, y) \ll \frac{c}{2}$. Hence we get that

$$d(x, y) \leq d(x, x_n) + d(x_n, y) \ll c.$$

Thus,

$$\lim \frac{c}{n} - d(x, y) = -d(x, y) \in \bar{P} = P.$$

So $d(x, y) \in P \cap (-P) = \{0\}$, i.e. $d(x, y) = 0$. $\diamondsuit$

Let X be a nonempty set and $f : X \to X$ an arbitrary mapping. $x \in X$ is a fixed point for f if $x = f(x)$. If $x_0 \in X$, we say that the sequence (x_n) defined by $x_n = f^n(x_0)$ is a sequence of Picard iterates of f at point x_0 or that (x_n) is the orbit of f at point x_0.

3 Contractive operators on complete F - spaces

We start with the following definition.

Definition 3.1 *If $A : E \to E$ is an one to one function such that $A(P) = P$, $(I - A)$ is one to one and $(I - A)(P) = P$ then A is contractive operator.*

Example 3.2 *Let n be a positive integer, $E = \mathbf{R}^n$, $P = \{(x_1, \ldots, x_n) \in E : x_i \geq 0 \ i = 1, \ldots, n\}$, $\lambda_1, \ldots, \lambda_n \in (0, 1)$ and $\Lambda = [a_{ij}]_{1 \leq i,j \leq n}$ be square matrix such that*

$$a_{ij} = \begin{cases} 0, & i \neq j; \\ \lambda_j, & i = j \end{cases}, 1 \leq i, j \leq n.$$

Then $A : E \to E$ defined by $A(x) = \Lambda[x]$ is contractive bounded linear operator.

Example 3.3 *Let* $E = \mathcal{C}^2([0,1])$ *with the norm*

$$\|f\| = \|f\|_\infty + \|f'\|_\infty,$$

and consider the cone

$$P = \{f \in E : f \geq 0\}.$$

Then P *is non-normal solid cone in* E *(see [11]).*

Then $A : E \to E$ *defined by*

$$A(f)|_x = \exp(-x)f(x)$$

is contractive bounded linear operator. We can see that $\|A\| = 3$*, and so* A *is not contractive operator in sense of Banach.*

Now we need the following Lemma.

Lemma 3.4 *If* $A : E \to E$ *is the contractive bounded linear operator then*

1) there exists A^{-1} *and it is bounded linear operator;*

2) there exists $(I - A)^{-1}$ *and it is bounded linear operator;*

3) $A(x) \ll x$ *for any* $x \in \mathrm{int}P$*;*

4) $x \leq y$ *implies* $A(x) \leq A(y)$ *for any* $x, y \in P$*;*

5) $x \ll y$ *implies* $A(x) \ll A(y)$ *for any* $x, y \in P$*;*

6) $(I - A)(x) \ll x$ *for any* $x \in \mathrm{int}P$*;*

7) $I + A + \cdots + A^n = (I - A)^{-1} \circ (I - A^{n+1})$*.*

Proof:

1) A^{-1} exists because A is one to one. For any $a, b \in E$ there exists $c, d \in E$ such that $a = A(c)$ and $b = A(d)$. From

$$A^{-1}(\alpha a+\beta b) = A^{-1}(\alpha A(c)+\beta A(d)) = A^{-1}(A(\alpha c+\beta d)) = \alpha A^{-1}(a)+\beta A^{-1}(b),$$

it follows that A^{-1} is linear. A is continuous because it is bounded, which implies that A^{-1} is continuous. So A^{-1} is bounded because it is linear.

2) $I - A$ is one to one bounded linear operator because I and A are one to one bounded linear operators. Now proof follows from 1).

3) From $(I - A)(P) = P$ by Open mapping theorem (see [12]) it follows that $(I - A)(\mathrm{int}P) \subseteq \mathrm{int}P$, which implies that $x - A(x) \in \mathrm{int}P$ for each $x \in \mathrm{int}P$.

4) From $x \leq y$ it follows $y - x \in P$ which implies $A(y - x) \in P$ because $A(P) = P$.

5) From $A(P) = P$ by Open mapping theorem (see [12]) it follows that $A(\text{int}P) \subseteq \text{int}P$. $x \ll y$ implies $y - x \in \text{int}P$ which implies $A(y-x) \in \text{int}P$ because $A(\text{int}P) \subseteq \text{int}P$.

6) It follows from $A(\text{int}P) \subseteq \text{int}P$ and $A(x) = x - (x - A(x))$.

7) It follows from $(I - A) \circ (I + A + \cdots + A^n) = I - A^{n+1}$. ♢

In this section our main result is the following theorem.

Theorem 3.5 *If $A : E \to E$ is the contractive bounded linear operator then for each $x \in P$ and any $c \in \text{int}P$ there exists a positive integer n_0 such that*

$$A^n(x) \ll c$$

for all $n > n_0$.

Proof: By Lemma 3.4 we get that

$$\begin{aligned}(I - A) \circ (n+1)A^n(x) \;\; &\leq \;\; (I - A) \circ (I + A + \cdots + A^n)(x) = \\ &= (I - A^{n+1})(x) = x - A^{n+1}(x) \leq x\end{aligned}$$

for any $x \in P$, because $A^n(x) \leq A^k(x)$ for any $k = 0, \ldots, n$. So

$$(I - A) \circ (n+1)A^n(x) \leq x.$$

Hence

$$A^n(x) \leq \frac{1}{n+1}(I - A)^{-1}(x).$$

For any $0 \ll c$ we get that there exists a positive integer n_0 such that $n > n_0$ implies

$$\frac{1}{n+1}(I - A)^{-1}(x) \ll c,$$

because

$$\frac{1}{n+1}(I - A)^{-1}(x)$$

is a convergent sequence. So $n > n_0$ implies

$$A^n(x) \ll c.$$

♢

From Lemma 3.4 and Theorem 3.5 we obtain:

Corollary 3.6 *If $A : E \to E$ is a contractive bounded linear operator then*

$$\lim_{n\to\infty}(I + A + \cdots + A^n) = (I - A)^{-1}.$$

Next Corollary extends famous Volterra's fixed point theorem (see [10]).

Corollary 3.7 *If $A : E \to E$ is a contractive bounded linear operator then for any $z \in P$ equation*

$$x = z + A(x)$$

has a unique solution $y \in P$ and

$$y = \lim_{n\to\infty} (I + A + \cdots + A^n)(x)$$

for any $x \in \mathrm{int}P$.

4 Main Results

Next Lemma is a partial generalization of a well known theorem of T. L. Hicks and B. E. Rhoades [5]. It also partially includes the recent result of M. Abbas and B. Rhoades [1] - Theorem 3.1.

Lemma 4.1 *Let (X, d) be an F - cone metric space, $(x_n) \subseteq X$ and $A : E \to E$ a contractive bounded linear operator. If*

$$d(x_{n+1}, x_{n+2}) \le A(d(x_n, x_{n+1})) \tag{2}$$

for any n then (x_n) is a Cauchy sequence.

Proof: For $m > n$ by (2) and Lemma 3.3 we obtain

$$\begin{aligned} &d(x_n, x_m) \le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \le \\ &\le (A^n + A^{n+1} + \cdots + A^{m-1})(d(x_0, x_1)) = \\ &= (I + A + \cdots + A^{m-n-1})(A^n(d(x_0, x_1))) \le (I - A)^{-1}(A^n(d(x_0, x_1))). \end{aligned}$$

By Theorem 3.5 it follows that (x_n) is a Cauchy sequence.$\diamondsuit$

Now we shall prove our main result, which generalizes a well known theorem of Ćiri'c [4].

Theorem 4.2 *Let (X, d) be a complete F - cone metric space, $f : X \to X$ and $A : E \to E$ a contractive bounded linear operator such that $A(P) = P$. If for any $x, y \in X$ there exists*

$$u \in \{d(x, y), d(x, f(x)), d(y, f(y)), \frac{d(x, f(y)) + d(y, f(x))}{2}\} \tag{3}$$

such that

$$d(f(x), f(y)) \le A(u), \tag{4}$$

then f has a unique fixed point $z \in X$ and for each $x \in X$ the sequence of Picard iterates defined by f at x converges to z.

Proof: Let $x_0 \in X$ be arbitrary and (x_n) the sequence of Picard iterates of f at point x_0. By (3) and (4) we get that there exists

$$u \in \{d(x_0,x_1), d(x_1,x_2), d(x_1,x_2), \frac{d(x_0,x_2)+d(x_1,x_1)}{2}\} \tag{5}$$

such that

$$d(x_1,x_2) \leq A(u). \tag{6}$$

Let $d(x_1,x_2) \not\leq A(d(x_0,x_1))$. From (5), (6) and Lemma 3.3 it follows that

$$d(x_1,x_2) \leq A(\frac{d(x_0,x_2)}{2}).$$

So

$$d(x_1,x_2) \leq A(d(x_0,x_2)) \leq \frac{A(d(x_0,x_1)) + A(d(x_1,x_2))}{2}$$

which implies

$$2d(x_1,x_2) \leq A(d(x_0,x_1)) + A(d(x_1,x_2)).$$

Hence

$$A(d(x_0,x_1)) + A(d(x_1,x_2)) - 2d(x_1,x_2) \in P,$$

which implies

$$A(d(x_0,x_1)) - d(x_1,x_2) \in P,$$

because $(I-A)(d(x_1,x_2)) \in P$. So

$$d(x_1,x_2) \leq A(d(x_0,x_1)),$$

which is a contradiction.

So (5) and (6) implies that

$$d(x_1,x_2) \leq A(d(x_0,x_1)). \tag{7}$$

From (7) by induction we obtain

$$d(x_{n+1},x_{n+2}) \leq A(d(x_n,x_{n+1})).$$

By Lemma 4.1 it follows that (x_n) is a Cauchy sequence. (x_n) is convergent because (X,d) is complete.

Let $\lim x_n = y$ and $y \neq f(y)$. Then there exists positive integer n_0 such that $n > n_0$ implies $d(x_n,y) \ll \frac{d(y,f(y))}{4}$. Let $n > n_0$. By (3) and (4) we get that there exists

$$u \in \{d(y,x_n), d(y,f(y)), d(x_n,x_{n+1}), \frac{d(y,x_{n+1})+d(x_n,f(y))}{2}\}$$

such that

$$d(f(y), x_{n+1}) \leq A(u).$$

So

$$d(y, f(y)) \leq d(f(y), x_{n+1}) + d(x_{n+1}, y) \leq A(u) + d(x_{n+1}, y).$$

Now we have the following possibilities.

1. $u = d(y, x_n)$ implies

$$d(y, f(y)) \leq d(f(y), x_{n+1}) + d(x_{n+1}, y) \leq A(d(y, x_n)) + d(x_{n+1}, y) \leq \frac{1}{2} d(y, f(y)),$$

which is a contradiction.

2. $u = d(y, f(y))$ implies $d(y, f(y)) \leq A(d(y, f(y)))$, which is a contradiction.

3. $u = d(x_n, x_{n+1})$ implies

$$d(y, f(y)) \leq d(x_n, y) + d(y, x_{n+1}) \leq \frac{1}{2} d(y, f(y)),$$

which is a contradiction.

4. $u = \dfrac{d(y, x_{n+1}) + d(x_n, f(y))}{2}$. Then

$$\begin{aligned} d(y, f(y)) &\leq A\left(\frac{d(y, x_{n+1}) + d(x_n, f(y))}{2}\right) + d(x_{n+1}, y) \\ &\leq A\left(\frac{d(y, x_{n+1}) + d(x_n, y) + d(y, f(y))}{2}\right) + d(x_{n+1}, y). \end{aligned}$$

From

$$2d(y, f(y)) - A(d(y, f(y))) \leq A(d(y, x_{n+1}) + d(x_n, y)) + d(x_{n+1}, y),$$

it follows that

$$d(y, f(y)) \leq d(y, x_{n+1}) + d(x_n, y) + d(x_{n+1}, y) \ll d(y, f(y)),$$

which is a contradiction.

So $y = f(y)$.

Let $z \in X$, $z \neq y$ and $z = f(z)$. From (3), (4) it follows that there exists

$$u \in \{d(z, y), d(z, z), d(y, y), \frac{d(z, y)) + d(z, y)}{2}\},$$

such that

$$d(f(z), f(y)) \leq A(u),$$

which implies $u = d(z, y)$ because $d(z, z) = d(y, y) = 0$. Hence

$$d(z, y) = d(f(z), f(y)) \leq A(d(z, y)),$$

which is a contradiction with Lemma 3.4. $\diamondsuit$

As a first application we present the next result which generalizes Corollary 2.4 of [11] and Corollary 1. of [6].

Corollary 4.3 *Let (X, d) be a complete F - cone metric space, $x_0 \in X$, $f : X \to X$, $0 \ll c$ and $A : E \to E$ a contractive bounded linear operator. Suppose that $d(x_0, f(x_0)) \leq (I - A)(c))$. If for any $x, y \in B(x_0, c)$ there exists*

$$u \in \{d(x, y), d(x, f(x)), d(y, f(y)), \frac{d(x, f(y)) + d(y, f(x))}{2}\}$$

such that

$$d(f(x), f(y)) \leq A(u),$$

then f has a unique fixed point $z \in B(x_0, c)$ and for each $x \in B(x_0, c)$ the sequence of Picard iterates defined by f at x converges to z.

Next Corollary generalizes Theorem 2.3 of [11] and Theorem 1. of [6]. This statement, when E is a Banach space, can also be obtained as corollary of Theorem 1 of [3].

Corollary 4.4 *Let (X, d) be a complete F - cone metric space, $f : X \to X$ and $A : E \to E$ a contractive bounded linear operator. If for any $x, y \in X$*

$$d(f(x), f(y)) \leq A(d(x, y)),$$

then f has a unique fixed point $z \in X$ and for each $x \in X$ the sequence of Picard iterates defined by f at x converges to z.

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