Results in Applied Mathematics 5 (2020) 100082

Contents lists available at ScienceDirect

Results in Applied Mathematics

journal homepage: www.elsevier.com/locate/results-in-applied-mathematics

Inverse problems for Sturm–Liouville differential operators with two constant delays under Robin boundary conditions

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ARTICLE INFO

Article history:
Received 24 June 2019
Received in revised form 21 September 2019
Accepted 25 October 2019
Available online 18 November 2019

MSC:
34B24
34A55

Keywords:
Differential operators with delays
Inverse spectral problems
Fourier coefficients

ABSTRACT

This paper deals with non-self-adjoint second-order differential operators with two constant delays from $[\pi/2, \pi)$ and two potentials from $L_2[0, \pi]$. We study the inverse spectral problem of recovering operators from their spectra. Four boundary value problems under Robin boundary conditions are considered. It has been proved that delays and the Fourier coefficients of potentials are uniquely determined from the spectra. Finally, potentials are constructed.

1. Introduction

The theory of differential equations with delays is a rapidly developing branch of the theory of ordinary differential equations. For a number of results revealing inverse spectral problems for classical Sturm–Liouville operators we refer the reader to [1]. However, some of the main methods in the inverse problem theory for classical Sturm–Liouville operators are not suitable for operators with delays. Although other effective methods have been created and some aspects of the direct and inverse problems for operators with a delay can be found in [2–10].

While there are a number results about both direct and inverse problems for operators with one delay, there are just a few results related to the operators with two or more delays [11–14]. The motivation behind this paper is to fill this gap and to initiate further research in the inverse spectral theory for differential operators with delays.

In what follows, we always take $k, i = 1, 2$.

In this paper we consider the boundary value problems $D_{1,i}$ of the form

$$-y''(x) + q_1(x)\, y(x-\tau_1) + q_2(x)\, y(x-\tau_2) = \lambda y(x), \, x \in [0, \pi] \quad (1)$$

$$y'(0) - h_i y(0) = 0 \quad (2)$$

$$y'(\pi) + Hy(\pi) = 0 \quad (3)$$

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https://doi.org/10.1016/j.rinam.2019.100082

where λ is a spectral parameter, $\frac{\pi}{2} \leq \tau_2 < \tau_1 < \pi$, $h_i, H \in R\backslash\{0\}$ and $q_i(x)$ are real-valued functions such that $q_i \in L_2[\tau_i, \pi]$ and $q_i(x) = 0, x \in [0, \tau_i)$. Besides the boundary value problems (1)–(3), we also consider boundary value problems $D_{2,i}$ of the form

$$-y''(x) + q_1(x)\, y(x - \tau_1) - q_2(x)\, y(x - \tau_2) = \lambda y(x), x \in [0, \pi] \tag{4}$$

under the same boundary conditions (2)–(3). We study the inverse spectral problem of recovering operators from the spectra of $D_{k,i}$. We use the method of Fourier coefficients, which based on determination of direct relations between Fourier coefficients of the potentials or some functions containing the potentials, and Fourier coefficients of some known functions.

Let $\left(\lambda_{n,i}^{(k)}\right)_{n=0}^{\infty}$ be the eigenvalues of $D_{k,i}$. The inverse problem is formulated as follows.

Inverse problem 1: Given $\left(\lambda_{n,i}^{(k)}\right)_{n=0}^{\infty}$, determine delays τ_i, parameters h_i and H and potentials q_i.

In Section 2, we study the spectral properties of the boundary value problems $D_{k,i}$. In Section 3, we prove that delays, parameters and the Fourier coefficients of potentials are uniquely determined from the spectra. Finally, potentials q_i are constructed.

2. Spectral properties

One can easily show that integral equation (1) under the initial condition (2) and conditions $q_i(x) = 0, x \in [0, \tau_i)$ is equivalent to the integral equation

$$y(x, z) = cosxz + \frac{h_i}{z} sinzx + \frac{1}{z}\sum_{k=1}^{2}\int_{\tau_k}^{x} q_k(t)\, sinz(x - t)\, y(t - \tau_k, z)\, dt. \tag{5}$$

Here and in the sequel $\lambda = z^2$. We solve the integral equation (5) by the method of steps.

Lemma 1. *The integral equation (5) on $(\tau_1, \pi]$ has the solution*

$$y(x, z) = coszx + \frac{h_i}{z} sinzx + \frac{1}{z}\sum_{k=1}^{2} b_{sc}^{(k)}(x, z) + \frac{h_i}{z^2}\sum_{i=1}^{2} b_{s^2}^{(k)}(x, z) \tag{6}$$

where

$$b_{sc}^{(k)}(x, z) = \int_{\tau_k}^{x} q_k(t)\, sinz(x - t)\, cosz(t - \tau_k)\, dt,$$

$$b_{s^2}^{(k)}(x, z) = \int_{\tau_k}^{x} q_k(t)\, sinz(x - t)\, sinz(t - \tau_k)\, dt.$$

Proof. It is obvious that for $x \in (0, \tau_2]$

$$y(x, z) = coszx + \frac{h_i}{z} sinzx.$$

For $x \in (\tau_2, \tau_1]$ we have

$$y(x, z) = coszx + \frac{h_i}{z} sinzx +$$
$$\frac{1}{z}\int_{\tau_2}^{x} q_2(t)\, sinz(x - t)\left(cosz(t - \tau_2) + \frac{h_i}{z} sinz(t - \tau_2)\right) dt =$$
$$coszx + \frac{h_i}{z} sinzx + \frac{1}{z} b_{sc}^{(2)}(x, z) + \frac{h_i}{z^2} b_{s^2}^{(2)}(x, z)$$

and for $x \in (\tau_1, \pi]$

$$y(x, z) = coszx + \frac{h_i}{z} sinzx +$$
$$\frac{1}{z}\sum_{k=1}^{2}\int_{\tau_k}^{x} q_k(t)\, sinz(x - t)\left(cosz(t - \tau_k) + \frac{h_i}{z} sinz(t - \tau_k)\right) dt =$$
$$coszx + \frac{h_i}{z} sinzx + \frac{1}{z}\sum_{k=1}^{2} b_{sc}^{(k)}(x, z) + \frac{h_i}{z^2}\sum_{k=1}^{2} b_{s^2}^{(k)}(x, z),$$

i.e. we get (6), thus proving Lemma 1. □

Denote $\Delta_{1,i}(\lambda) = F_i(z) = y'(\pi, z) + Hy(\pi, z)$. From (6) we obtain

$$\Delta_{1,i}(\lambda) = F_i(z) = \left(-z + \frac{h_i H}{z}\right) sin\pi z + (h_i + H)\, cos\pi z + \sum_{k=1}^{2} b_{c^2}^{(k)}(z) + \frac{h_i}{z}\sum_{k=1}^{2} b_{cs}^{(k)}(z) + \frac{H}{z}\sum_{k=1}^{2} b_{sc}^{(k)}(z) + \frac{h_i H}{z^2}\sum_{k=1}^{2} b_{s^2}^{(k)}(z) \tag{7}$$

where

$$b_{cs}^{(k)}(z) = \int_{\tau_k}^{\pi} q_k(t)\, cosz(\pi - t)\, sinz(t - \tau_k)\, dt,$$
$$b_{c^2}^{(k)}(z) = \int_{\tau_k}^{\pi} q_k(t)\, cosz(\pi - t)\, cosz(t - \tau_k)\, dt.$$

Here and in the sequel we write $b_{sc}^{(k)}(z)$ instead of $b_{sc}^{(k)}(\pi, z)$ and $b_{s^2}^{(k)}(z)$ instead of $b_{s^2}^{(k)}(\pi, z)$. The functions $\Delta_{1,i}(\lambda)$ are entire in λ of order $1/2$. Taking (3) into account, it is obvious that the zeros of $\Delta_{1,i}(\lambda)$coincide with the eigenvalues $\left(\lambda_{n,i}^{(1)}\right)_{n=0}^{\infty}$ of $D_{1,i}$. Therefore, the functions $\Delta_{1,i}(\lambda)$ are the characteristic functions for $D_{1,i}$.

In order to solve the inverse problem, we should transform the characteristic functions (7). For this purpose, let us define the transitional function $\tilde{q}$

$$\tilde{q}(t) = \begin{cases} q_1\left(t + \frac{\tau_1}{2}\right) + q_2\left(t + \frac{\tau_2}{2}\right), t \in \left[\frac{\tau_1}{2}, \pi - \frac{\tau_1}{2}\right] \\ q_2\left(t + \frac{\tau_2}{2}\right), t \in \left[\frac{\tau_2}{2}, \frac{\tau_1}{2}\right) \cup \left(\pi - \frac{\tau_1}{2}, \pi - \frac{\tau_2}{2}\right] \\ 0, t \in \left[0, \frac{\tau_2}{2}\right) \cup \left(\pi - \frac{\tau_2}{2}, \pi\right]. \end{cases} \tag{8}$$

Denote

$$J_1^{(k)} = \int_{\tau_k}^{\pi} q_k(t)\, dt$$

and

$$\tilde{a}_c(z) = \int_0^{\pi} \tilde{q}(t)\, cosz(\pi - 2t)dt, \quad \tilde{a}_s(z) = \int_0^{\pi} \tilde{q}(t)\, sinz(\pi - 2t)dt.$$

One can easily show that the following relations hold:

$$\sum_{k=1}^{2} b_{c^2}^{(k)}(z) = \frac{1}{2}\left(\tilde{a}_c(z) + J_{1,c}(z)\right), \sum_{k=1}^{2} b_{s^2}^{(k)}(z) = \frac{1}{2}\left(\tilde{a}_c(z) - J_{1,c}(z)\right),$$
$$\sum_{k=1}^{2} b_{cs}^{(k)}(z) = \frac{1}{2}\left(-\tilde{a}_s(z) + J_{1,s}(z)\right), \sum_{k=1}^{2} b_{sc}^{(k)}(z) = \frac{1}{2}\left(\tilde{a}_s(z) + J_{1,s}(z)\right) \tag{9}$$

where

$$J_{1,c}(z) = \sum_{k=1}^{2} J_1^{(k)} cosz(\pi - \tau_k), J_{1,s}(z) = \sum_{k=1}^{2} J_1^{(k)} sinz(\pi - \tau_k)$$

Substituting (9) into (7), we rewrite the characteristic functions as follows:

$$\Delta_{1,i}(\lambda) = F_i(z) = \left(-z + \frac{h_i H}{z}\right) sin\pi z + (h_i + H)\, cos\pi z + \frac{1}{2}\left(\tilde{a}_c(z) + J_{1,c}(z)\right) + \frac{h_i}{2z}\left(-\tilde{a}_s(z) + J_{1,s}(z)\right) + \frac{H}{2z}\left(\tilde{a}_s(z) + J_{1,s}(z)\right) + \frac{h_i H}{2z^2}\left(\tilde{a}_c(z) - J_{1,c}(z)\right). \tag{10}$$

Using (10), by the well known method (see [1]),we obtain the asymptotic formulas for the eigenvalues $\lambda_{n,i}^{(1)}$:

$$\lambda_{n,i}^{(1)} = n^2 + \frac{2}{\pi}(h_i + H) + \frac{J_1^{(1)}}{\pi} cosn\tau_1 + \frac{J_1^{(2)}}{\pi} cosn\tau_2 + o(1), n \to \infty. \tag{11}$$

In the same way we consider the boundary value problem $D_{2,i}$. First, let us define the transitional function $\tilde{Q}$

$$\tilde{Q}(t) = \begin{cases} q_1\left(t + \frac{\tau_1}{2}\right) - q_2\left(t + \frac{\tau_2}{2}\right), t \in \left[\frac{\tau_1}{2}, \pi - \frac{\tau_1}{2}\right] \\ -q_2\left(t + \frac{\tau_2}{2}\right), t \in \left[\frac{\tau_2}{2}, \frac{\tau_1}{2}\right) \cup \left(\pi - \frac{\tau_1}{2}, \pi - \frac{\tau_2}{2}\right] \\ 0, t \in \left[0, \frac{\tau_2}{2}\right) \cup \left(\pi - \frac{\tau_2}{2}, \pi\right] \end{cases} \tag{12}$$

and denote:

$$\tilde{A}_c(z) = \int_0^\pi \tilde{Q}(t)\, cosz(\pi - 2t)dt, \tilde{A}_s(z) = \int_0^\pi \tilde{Q}(t)\, sinz(\pi - 2t)dt. \tag{13}$$

Then, one can show that the characteristic functions for $D_{2,i}$ has the form

$$\Delta_{2,i}(\lambda) = G_i(z) = \left(-z + \frac{h_i H}{z}\right) sin\pi z + (h_i + H)\, cos\pi z + \frac{1}{2}\left(\tilde{A}_c(z) + J^*_{1,c}(z)\right) +$$
$$\frac{h_i}{2z}\left(-\tilde{A}_s(z) + J^*_{1,s}(z)\right) + \frac{H}{2z}\left(\tilde{A}_s(z) + J^*_{1,s}(z)\right) + \frac{h_i H}{2z^2}\left(\tilde{A}_c(z) - J^*_{1,c}(z)\right)$$

where

$$J^*_{1,c}(z) = J_1^{(1)} cosz(\pi - \tau_1) - J_1^{(2)} cosz(\pi - \tau_2),$$
$$J^*_{1,s}(z) = J_1^{(1)} sinz(\pi - \tau_1) - J_1^{(2)} sinz(\pi - \tau_2).$$

At the end, using (13) we obtain the asymptotic formulas for the eigenvalues $\lambda^{(2)}_{n,i}$:

$$\lambda^{(2)}_{n,i} = n^2 + \frac{2}{\pi}(h_i + H) + \frac{J_1^{(1)}}{\pi} cosn\tau_1 - \frac{J_1^{(2)}}{\pi} cosn\tau_2 + o(1), n \to \infty. \tag{14}$$

Now, from the spectra of $D_{k,i}$ we can construct the characteristic functions $\Delta_{k,i}(\lambda)$. By Hadamard's factorization theorem (see [15]), characteristic function is uniquely determined up to a multiplicative constant by its zeros. Then next lemma holds.

Lemma 2. *The specifications of spectrum* $\left(\lambda^{(k)}_{n,i}\right)_{n=0}^{\infty}$ *uniquely determine the characteristic functions* $\Delta_{k,i}(\lambda)$ *by the formulas*

$$\Delta_{k,i}(\lambda) = \pi\left(\lambda^{(k)}_{0,i} - \lambda\right)\prod_{n=1}^{\infty}\frac{\lambda^{(k)}_{n,i} - \lambda}{n^2}. \tag{15}$$

3. Main results

At the beginning, we show that delays are uniquely determined from the spectra of $D_{1,i}$.

Lemma 3. *Delays* τ_i *are uniquely determined by eigenvalues* $\left(\lambda^{(1)}_{n,i}\right)_{n=0}^{\infty}$ *of* $D_{1,i}$.

Proof. Let us consider the sequences

$$\rho_{n,i} = \frac{\lambda^{(1)}_{n,i} + \lambda^{(2)}_{n,i}}{2}$$

and

$$\sigma_n = \frac{\lambda^{(1)}_{n,i} - \lambda^{(2)}_{n,i}}{2}.$$

From (11) and (14) we obtain the asymptotic formulas:

$$\rho_{n,i} = n^2 + \frac{2}{\pi}(h_i + H) + \frac{J_1^{(1)}}{\pi} cosn\tau_1 + o(1), n \to \infty \tag{16}$$

and

$$\sigma_n = \frac{J_1^{(2)}}{\pi} cosn\tau_2 + o(1), n \to \infty. \tag{17}$$

Let us consider the sequence $(\psi_n)_{n=0}^{\infty}$

$$\psi_n = \frac{\sigma_{n-2} - \sigma_{n+2}}{\sigma_{n-1} - \sigma_{n+1}}.$$

Using the asymptotic formula (17) for $n \to \infty$ we get

$$\psi_n = \frac{\cos(n-2)\tau_2 - \cos(n+2)\tau_2 + o(1)}{\cos(n-1)\tau_2 - \cos(n+1)\tau_2 + o(1)} = \frac{sinn\tau_2 \sin 2\tau_2 + o(1)}{sinn\tau_2 \sin\tau_2 + o(1)} \to 2\cos\tau_2$$

i.e.

$$\tau_2 = \arccos\left(\frac{1}{2}\lim_{n\to\infty}\psi_n\right). \tag{18}$$

In order to determine τ_1, let us consider the sequence $(\varphi_{n,i})_{n=0}^{\infty}$

$$\varphi_{n,i} = \frac{\rho_{n-2,i} - (n-2)^2 - \rho_{n+2,i} + (n+2)^2}{\rho_{n-1,i} - (n-1)^2 - \rho_{n+1,i} + (n+1)^2}.$$

Using the asymptotic formulas (16) for $n \to \infty$ we get

$$\varphi_{n,i} = \frac{\cos(n-2)\tau_1 - \cos(n+2)\tau_1 + o(1)}{\cos(n-1)\tau_1 - \cos(n+1)\tau_1 + o(1)} = \frac{sinn\tau_1 \sin 2\tau_1 + o(1)}{sinn\tau_1 \sin\tau_1 + o(1)} \to 2\cos\tau_1.$$

Then

$$\tau_1 = \arccos\left(\frac{1}{2}\lim_{n\to\infty}\varphi_{n,i}\right)$$

thus proving Lemma 3. □

Lemma 4. *Integrals $J_1^{(i)}$ are uniquely determined by eigenvalues $\left(\lambda_{n,i}^{(1)}\right)_{n=0}^{\infty}$ of $D_{1,i}$.*

Proof. Let us select the subsequence (n_k) such that $(\forall k)\ |cosn_k\tau_2| > 0$. Taking (17) into account, we obtain

$$J_1^{(2)} = \pi \lim_{k\to\infty}\frac{\sigma_{n_k}}{\cos n_k\tau_2}.$$

Now we determine $h_i + H$. For this purpose, let us select subsequences (n_k) and (m_k) such that $(\forall k)\ \left(cosn_k\tau_1 cosm_k\tau_1 \neq 0 \bigwedge |cosn_k\tau_1 - cosm_k\tau_1| > 0\right)$. Then from (18) we get

$$h_i + H = \frac{\pi}{2}\lim_{k\to\infty}\frac{\left(\rho_{m_k,i} - (m_k)^2\right)\cos n_k\tau_1 - \left(\rho_{n_k,i} - (n_k)^2\right)\cos m_k\tau_1}{\cos n_k\tau_1 - \cos m_k\tau_1}. \tag{19}$$

Now, if we select the subsequence (n_k) such that $(\forall k)\ |cosn_k\tau_1| > 0$, it follows from (18)

$$J_1^{(1)} = \pi \lim_{k\to\infty}\frac{\rho_{n_k,i} - (n_k)^2 - \frac{2}{\pi}(h_i + H)}{\cos n_k\tau_1}.$$

Lemma 4 is proved. □

Lemma 5. *Parameters H and h_i are uniquely determined by eigenvalues $\left(\lambda_{n,i}^{(1)}\right)_{n=0}^{\infty}$ of $D_{1,i}$.*

Proof. Note that $h_i + H$ are determined by (19), so $h_2 - h_1$ is known, too. Further, it follows from Lemma 2 that characteristic functions are uniquely determined by the spectrum, which means that characteristic functions $\Delta_{1,i}(\lambda)$ have the form (10). Taking into account that $J_{1,c}(z)$ and $J_{1,s}(z)$ by virtue of Lemma 3 and Lemma 4 are also known, putting $\lambda = \left(\frac{4k+1}{2}\right)^2$ into (15), we can define functions

$$F_i^*(k) = \Delta_{1,i}\left(\left(\frac{4k+1}{2}\right)^2\right) + \frac{4k+1}{2} - \frac{1}{2}\ J_{1,c}\left(\tfrac{4k+1}{2}\right)\ - \frac{h_i + H}{4k+1}J_{1,s}\left(\frac{4k+1}{2}\right).$$

Then from (10) we obtain

$$\frac{4k+1}{h_2 - h_1}\left(F_2^*(k) - F_1^*(k)\right) = 2H - \tilde{a}_s\left(\frac{4k+1}{2}\right) + \frac{2H}{4k+1}\left(\tilde{a}_c\left(\frac{4k+1}{2}\right) - J_{1,c}\left(\frac{4k+1}{2}\right)\right).$$

Since

$$\tilde{a}_s\left(\frac{4k+1}{2}\right) = \int_0^{\pi}\tilde{q}(t)\cos(4k+1)t\,dt = o(1),\ k\to\infty,$$

we obtain

$$H = \frac{1}{2}\lim_{k\to\infty}\frac{4k+1}{h_2 - h_1}\left(F_2^*(k) - F_1^*(k)\right).$$

At the end we determine h_i from $h_i + H$, thus proving Lemma 5. □

Finally, we come to our main result. Denote:

$$\tilde{a}_{2n} = \int_0^{\pi} \tilde{q}\,(t)\cos 2nt dt, \quad \tilde{b}_{2n} = \int_0^{\pi} \tilde{q}\,(t)\sin 2nt dt,$$

and

$$\tilde{A}_{2n} = \int_0^{\pi} \tilde{Q}\,(t)\cos 2nt dt, \quad \tilde{B}_{2n} = \int_0^{\pi} \tilde{Q}\,(t)\sin 2nt dt.$$

The next Theorem holds.

Theorem 1. *The Fourier coefficients $\frac{2}{\pi}\tilde{a}_{2n}, n \in N_0$ and $\frac{2}{\pi}\tilde{b}_{2n}, n \in N$ of the transitional function $\tilde{q}$ are uniquely determined by eigenvalues $\left(\lambda_{n,i}^{(1)}\right)_{n=0}^{\infty}$ of $D_{1,i}$.*

Proof. Let us define the functions

$$A\,(z) = \frac{2}{h_2 - h_1}\left(h_2 F_1\,(z) - h_1 F_2\,(z)\right) + 2z\sin\pi z - 2H\cos\pi z - J_{1,c}\,(z) - \frac{H}{z} J_{1,s}(z)$$

and

$$B\,(z) = \frac{2z}{h_2 - h_1}\left(F_2\,(z) - F_1\,(z)\right) - 2H\sin\pi z - 2z\cos\pi z - J_{1,s}\,(z) + \frac{H}{z} J_{1,c}\,(z)\,.$$

Taking (10) into account, we obtain

$$\begin{aligned} A\,(z) &= \tilde{a}_c\,(z) + \frac{H}{z}\tilde{a}_s\,(z) \\ B\,(z) &= \frac{H}{z}\tilde{a}_c\,(z) - \tilde{a}_s\,(z)\,. \end{aligned} \tag{20}$$

Putting $z = n, n \in N$ into (20), we obtain the system

$$\begin{aligned} (-1)^n A\,(n) &= \tilde{a}_{2n} - \frac{H}{n}\tilde{b}_{2n} \\ (-1)^n B\,(n) &= \frac{H}{n}\tilde{a}_{2n} + \tilde{b}_{2n}. \end{aligned} \tag{21}$$

The system (21) has the unique solution

$$\begin{aligned} \tilde{a}_{2n} &= (-1)^n \frac{n}{H^2 + n^2}\left(nA\,(n) + HB\,(n)\right) \\ \tilde{b}_{2n} &= (-1)^n \frac{n}{H^2 + n^2}\left(nB\,(n) - HA\,(n)\right), n \in N. \end{aligned}$$

Then the Fourier coefficients $\frac{2}{\pi}\tilde{a}_{2n}$ and $\frac{2}{\pi}\tilde{b}_{2n}, n \in N$ of the transitional function $\tilde{q}$ are uniquely determined. Since

$$\tilde{a}_0 = \int_0^{\pi} \tilde{q}\,(t)\,dt = \int_{\frac{\tau_1}{2}}^{\pi - \frac{\tau_1}{2}} q_1\left(t + \frac{\tau_1}{2}\right)dt + \int_{\frac{\tau_2}{2}}^{\pi - \frac{\tau_2}{2}} q_2\left(t + \frac{\tau_2}{2}\right)dt = J_1^{(1)} + J_1^{(2)}$$

by virtue of Lemma 4 we get that the Fourier coefficient $\frac{2}{\pi}\tilde{a}_0$ of the transitional function $\tilde{q}$ is also uniquely determined. Theorem is proved. □

Since $\tilde{q} \in L_2\,[0, \pi]$, by virtue of Theorem 1, the uniqueness of the transitional function $\tilde{q}$ is proved. Then, we construct $\tilde{q}$ by

$$\tilde{q}\,(t) = \frac{\tilde{a}_0}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty} \tilde{a}_{2n}\cos 2nt + \tilde{b}_{2n}\sin 2nt,\, t \in \left[\frac{\tau_2}{2}, \pi - \frac{\tau_2}{2}\right]. \tag{22}$$

In the same way we show that Fourier coefficients $\frac{2}{\pi}\tilde{A}_{2n}, n \in N_0$ and $\frac{2}{\pi}\tilde{B}_{2n}, n \in N$ of the transitional function $\tilde{Q}$ are uniquely determined by eigenvalues $\left(\lambda_{n,i}^{(2)}\right)_{n=0}^{\infty}$ of $D_{2,i}$ and then we obtain

$$\tilde{Q}\,(t) = \frac{\tilde{A}_0}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty} \tilde{A}_{2n}\cos 2nt + \tilde{B}_{2n}\sin 2nt,\, t \in \left[\frac{\tau_2}{2}, \pi - \frac{\tau_2}{2}\right]. \tag{23}$$

Now from (8), (12), (22) and (23) we obtain

$$q_1\left(t + \frac{\tau_1}{2}\right) = \frac{\tilde{q}\,(t) + \tilde{Q}\,(t)}{2},\, t \in \left[\frac{\tau_1}{2}, \pi - \frac{\tau_1}{2}\right]$$

and

$$q_2\left(t+\frac{\tau_2}{2}\right)=\frac{\tilde{q}(t)-\tilde{Q}(t)}{2},\, t\in\left[\frac{\tau_2}{2},\pi-\frac{\tau_2}{2}\right].$$

Hence

$$q_1(x)=\frac{1}{2}\left(\tilde{q}\left(x-\frac{\tau_1}{2}\right)+\tilde{Q}\left(x-\frac{\tau_1}{2}\right)\right),\, x\in[\tau_1,\pi] \tag{24}$$

and

$$q_2(x)=\frac{1}{2}\left(\tilde{q}\left(x-\frac{\tau_2}{2}\right)-\tilde{Q}\left(x-\frac{\tau_2}{2}\right)\right),\, x\in[\tau_2,\pi]. \tag{25}$$

The potentials q_1 and q_2 are uniquely determined by (24) and (25), and Inverse problem 1 is solved.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

Funds for the publication of the paper were provided by the Ministry of Scientific and Technological Development, Higher Education and Information Society of Republic of Srpska.

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