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Asymptotic similarity relation and generalized inverse

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Abstract. In this paper we discuss relationships among asymptotic similarity, weak asymptotic equivalence and the generalized inverse in the class $\mathcal{A}$ of all nondecreasing unbounded positive functions on a half-axis $[a,+\infty)$ $(a>0)$. As a main result, we prove proper characterizations of some classes of functions in Karamata's theory of regular and rapid variation.

1. Introduction

Karamata's theory of regular variability (see e.g. [15]) is obtained from the serious research of Tauberian type problems (see e.g. [16]). Soon after, it becomes a very important part of asymptotic analysis, with many applications in other fields of mathematics (see e.g. [3]). The main object in Karamata's theory of regular variability is the class of O-regularly varying functions (class ORV).

A function $f:[a,+\infty)\to(0,+\infty)$ $(a>0)$ belongs to the class ORV if it is measurable and satisfy the following asymptotic condition (so-called Tauberian condition, [1]):

$$\underline{k}_f(\lambda)=\underline{\lim}_{x\to+\infty}\frac{f(\lambda x)}{f(x)}>0, \tag{1}$$

for every $\lambda>0$.

A function $f\in ORV$ is *regularly varying in the sense of Karamata* if $\underline{k}_f(\lambda)$ is differentiable for all $\lambda>0$, thus if there is a $\rho\in\mathbb{R}$ such that

$$\underline{k}_f(\lambda)=\lambda^\rho \tag{2}$$

for every $\lambda>0$; then ρ is called *the index of variability of f*.

The class of all regularly varying functions in the sense of Karamata is denoted RV. Any function $f\in RV$ having index of variability $\rho=0$ is called *slowly varying in the sense of Karamata*. The class of slowly varying functions in the sense of Karamata is denoted SV. It is well-known that $SV\subsetneq RV\subsetneq ORV$, and classes SV and RV represent the most utilized objects of Karamata's theory of regular variability (see, e.g. [17]).

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A function $f : [a, +\infty) \to (0, +\infty)$ for some $a > 0$ is called *positively increasing* (i.e. belongs to the class *PI*) if it is measurable and there is a $\lambda_0 > 1$ such that

$$\underline{k}_f(\lambda_0) > 1. \tag{3}$$

A function $f : [a, +\infty) \to (0, +\infty)$, $a > 0$, is called *rapidly varying in the sense of de Haan* with index $+\infty$ (i.e. belongs to the class R_∞) if it is measurable and

$$\underline{k}_f(\lambda) = +\infty \tag{4}$$

for every $\lambda > 1$.

We have $R_\infty \subsetneq PI$. The classes R_∞ and *PI* are important objects in quantitative analysis of divergent processes of moderate and rapid increase (see [5, 7, 14]).

Let f and g be two positive functions on $[a, +\infty)$, $a > 0$. They are called *weakly asymptotically equivalent* and denoted $f(x) \asymp g(x)$ as $x \to +\infty$, if

$$0 < \underline{\lim}_{x\to+\infty} \frac{f(x)}{g(x)} \leq \overline{\lim}_{x\to+\infty} \frac{f(x)}{g(x)} < +\infty. \tag{5}$$

Let $f : [a, +\infty) \to (0, +\infty)$, $a > 0$, and let $\{f\} = \{g : [a, +\infty) \to (0, +\infty) \mid f(x) \asymp g(x),\ \text{as}\ x \to +\infty\}$. Then $g \in \{f\}$ is called *asymptotically similar to f* if

$$0 < \underline{\lim}_{x\to+\infty} \frac{f(x)}{g(x)} = \overline{\lim}_{x\to+\infty} \frac{f(x)}{g(x)} < +\infty, \tag{6}$$

which is denoted by $f(x) \bowtie g(x)$, as $x \to +\infty$.

Let $f : [a, +\infty) \to (0, +\infty)$, $a > 0$, and let $\langle f\rangle = \{g \in \{f\} \mid f(x) \bowtie g(x),\ \text{for}\ x \to +\infty\}$. If $f : [a, +\infty) \to (0, +\infty)$, for some $a > 0$, define

$$[f]_\sim = \{g : [a, +\infty) \to (0, +\infty) \mid f(x) \sim g(x),\ x \to +\infty\},$$

where $f(x) \sim g(x)$, $x \to +\infty$, is the *strong asymptotic equivalence relation* defined by

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = 1.$$

Then $[f]_\sim \subsetneq \langle f\rangle \subsetneq \{f\}$.

In this paper we shall discuss the class of functions $\mathcal{A} = \{f : [a, +\infty) \to (0, +\infty)$, for some $a > 0 \mid f$ nondecreasing and unbounded$\}$, as well as the operator $f^{\leftarrow}(x) = \inf\{y > a \mid f(y) > x\}$ ($f \in \mathcal{A}$, $x \geq f(a)$).

2. Main results

The next results are continuation of research published in [2, 8–12].

Proposition 2.1. *Let $f \in \mathcal{A}$. Also, let $f \in R_\infty$, or f is regularly varying function in the sense of Karamata with index of variability $\rho > 0$. Then $g^{\leftarrow} \in \langle f^{\leftarrow}\rangle$ for every $g \in \mathcal{A} \cap \langle f\rangle$.*

Proof. First assume $f \in \mathcal{A} \cap RV_\rho$ for some $\rho > 0$ and let $g \in \mathcal{A} \cap \langle f\rangle$. Then there is an $\alpha = \alpha(g) > 0$ such that $\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \alpha$. For any $\varepsilon > 1$ we have

$$\frac{\alpha}{\varepsilon} \leq \frac{f(x)}{g(x)} \leq \varepsilon \cdot \alpha$$

for all sufficiently large x. Hence we obtain

$$f^{\leftarrow}(x) \geq g^{\leftarrow}\Big(\frac{x}{\varepsilon \cdot \alpha}\Big) \qquad \text{and} \qquad f^{\leftarrow}(x) \leq g^{\leftarrow}\Big(\frac{\varepsilon}{\alpha} \cdot x\Big),$$

for all sufficiently large x.

For the same x we have

$$\frac{g^{\leftarrow}(\frac{x}{\varepsilon \cdot \alpha})}{g^{\leftarrow}(x)} \leq \frac{f^{\leftarrow}(x)}{g^{\leftarrow}(x)} \leq \frac{g^{\leftarrow}(\frac{\varepsilon}{\alpha} \cdot x)}{g^{\leftarrow}(x)}.$$

Since $g \in \langle f \rangle$, we find $g \in RV_\rho$ and we obtain

$$(\varepsilon \cdot \alpha)^{-1/\rho} \leq \varliminf_{x \to +\infty} \frac{f^{\leftarrow}(x)}{g^{\leftarrow}(x)} \leq \varlimsup_{x \to +\infty} \frac{f^{\leftarrow}(x)}{g^{\leftarrow}(x)} \leq \Big(\frac{\varepsilon}{\alpha}\Big)^{1/\rho}.$$

If in the previous inequalities we let $\varepsilon \to 1_+$, it follows that $\lim_{x \to +\infty} \frac{f^{\leftarrow}(x)}{g^{\leftarrow}(x)} = \alpha^{-1/\rho}$, so that $g^{\leftarrow} \in \langle f^{\leftarrow} \rangle$.

Next let $f \in \mathcal{A} \cap R_\infty$ and $g \in \mathcal{A} \cap \langle f \rangle$. Then $g \in \{f\}$ and by some results from [10] it follows that $\lim_{x \to +\infty} \frac{f^{\leftarrow}(x)}{g^{\leftarrow}(x)} = 1$, i.e. $g^{\leftarrow} \in \langle f^{\leftarrow} \rangle$. □

Proposition 2.2. *If $f \in \mathcal{A}$ and $g^{\leftarrow} \in \langle f^{\leftarrow} \rangle$ for every $g \in \mathcal{A} \cap \langle f \rangle$, then $f \in R_\infty \cup \bigcup_{\rho > 0} R_\rho$. The same conclusion holds for every $g \in \langle f \rangle$.*

Proof. Let $f \in \mathcal{A}$ and let $g_1(x) = \alpha \cdot f(x)$ for $x \geq a$, where $\alpha > 0$ is arbitrary fixed number. Since $g_1 \in \mathcal{A} \cap \langle f \rangle$, it follows that $g_1^{\leftarrow} \in \langle f^{\leftarrow} \rangle$, i.e. there is a $\beta = \beta(\alpha) \in (0, +\infty)$ such that

$$\lim_{x \to +\infty} \frac{f^{\leftarrow}(\frac{1}{\alpha} \cdot x)}{f^{\leftarrow}(x)} = \lim_{x \to +\infty} \frac{g_1^{\leftarrow}(x)}{f^{\leftarrow}(x)} = \beta.$$

Since $f^{\leftarrow} \in \mathcal{A}$, it follows that $f^{\leftarrow} \in RV_\rho$ for some $\rho \geq 0$. If $\rho = 0$, then by [13] we get $f \in R_\infty$. If $\rho > 0$, then $f \in RV_{1/\rho}$ (the well-known result which can be found in [3]). Next, let $g \in \mathcal{A} \cap \langle f \rangle$ be fixed. Then

$$\lim_{x \to +\infty} \frac{g(\gamma x)}{g(x)} = \lim_{x \to +\infty} \frac{f(\gamma x)}{f(x)},$$

for every $\gamma > 0$, and hence $g \in R_\infty \cup RV_{1/\rho}$ for some $\rho \in (0, +\infty)$. □

Proposition 2.3. *If $f \in \mathcal{A} \cap RV$ with index of variability $\rho \geq 0$, theng $\in \langle f \rangle$ whenever $g \in \mathcal{A}$ and $g^{\leftarrow} \in \langle f^{\leftarrow} \rangle$.*

Proof. Let $f \in \mathcal{A} \cap RV_\rho$ for some $\rho \geq 0$. Then $f^{\leftarrow} \in R_\infty$, or $f^{\leftarrow}$ is regularly varying in the sense of Karamata with positive index of variability (see, e.g. [3]). Next, assume $g \in \mathcal{A}$ and $g^{\leftarrow} \in \langle f^{\leftarrow} \rangle$. Since $g^{\leftarrow} \in \mathcal{A}$ and $f^{\leftarrow} \in \mathcal{A}$, Proposition 2.1 yields $(f^{\leftarrow}(x))^{\leftarrow} \asymp (g^{\leftarrow}(x))^{\leftarrow}$ as $x \to +\infty$. Since $f \in RV_\rho$ for $\rho \geq 0$, and $f \in \mathcal{A}$, by [12] we immediately obtain $\lim_{x \to +\infty} \frac{(f^{\leftarrow}(x))^{\leftarrow}}{f(x)} = 1$. Since $g^{\leftarrow} \in \langle f^{\leftarrow} \rangle$, we have that $g^{\leftarrow} \in R_\infty$, or $g^{\leftarrow}$ is regularly varying in the sense of Karamata with positive index of variability and $g \in RV_\rho$ for some $\rho \geq 0$. Hence $\lim_{x \to +\infty} \frac{(g^{\leftarrow}(x))^{\leftarrow}}{g(x)} = 1$, and we finally obtain $g \in \langle f \rangle$. □

Proposition 2.4. *If $f \in \mathcal{A}$ and $g \in \langle f \rangle$ whenever $g \in \mathcal{A}$ and $g^{\leftarrow} \in \langle f^{\leftarrow} \rangle$, then $f \in RV$ with index of variability $\rho \geq 0$. The same conclusion holds for every $g \in \langle f \rangle$.*

Proof. Assume $f \in \mathcal{A}$ and define $g_1(x) = f(\alpha x)$, for $x \geq a$, where $\alpha > 1$ is an arbitrary fixed number. We find that $g_1^{\leftarrow}(x) = \dfrac{1}{\alpha} \cdot f^{\leftarrow}(x)$ for all sufficiently large x, and hence $g_1^{\leftarrow} \in \langle f^{\leftarrow} \rangle$, i.e. $g_1 \in \langle f \rangle$. Therefore,

$$\lim_{x\to+\infty} \frac{g_1(x)}{f(x)} = \lim_{x\to+\infty} \frac{f(\alpha x)}{f(x)} = \beta,$$

for some $\beta \in (0, +\infty)$, where β is a function of $\alpha > 1$. Accordingly, $f \in RV_\rho$ for some $\rho \geq 0$. Next, let $g \in \mathcal{A}$ be an arbitrary and fixed function and suppose $g \in \langle f \rangle$. Then analogously to the proof of Proposition 2.2 one can prove that $g \in RV_\rho$ for some $\rho \geq 0$. □

Combining the results from Propositions 2.1, 2.2, 2.3 and 2.4, the next corollary can be obtained. Notice that it can be also obtained by some results in [4] and [6].

Corollary 2.5. *Let $f \in \mathcal{A} \cap RV$ with index of variability $\rho > 0$. Then $g \in \langle f \rangle$ for every $g \in \mathcal{A}$ if and only if $g^{\leftarrow} \in \langle f^{\leftarrow} \rangle$. If $f \in \mathcal{A}$ and f is not regularly varying in the sense of Karamata with index of variability $\rho > 0$, then there is a $g \in \mathcal{A}$ such that $g \in \langle f \rangle$ and $g^{\leftarrow} \notin \langle f^{\leftarrow} \rangle$, or $g^{\leftarrow} \in \langle f^{\leftarrow} \rangle$ and $g \notin \langle f \rangle$.*

By this corollary we will prove Proposition 2.6, Proposition 2.8, Corollary 2.9 and Corollary 2.10.

Proposition 2.6. *Let $f \in \mathcal{A}$.*

(a) If $f \in R_\infty$ then $g^{\leftarrow} \in [f^{\leftarrow}]_\sim \subsetneq \langle f^{\leftarrow} \rangle$ for every $g \in \{f\}$;

(b) If $g^{\leftarrow} \in \langle f^{\leftarrow} \rangle$ whenever $g \in \mathcal{A} \cap \{f\}$, then $f \in RV_\rho \cup R_\infty$ for some $\rho > 0$;.

(c) If $f \in RV_\rho$, $\rho > 0$ then there is a $g \in \mathcal{A}$ such that $g \in \{f\}$ and $g^{\leftarrow} \notin \langle f^{\leftarrow} \rangle$.

Proof. (a) Follows from some results in [10].

(b)According to assumptions and Proposition 2.2 we have $f \in RV_\rho \cup R_\infty$ for some $\rho > 0$.

(c) Assume that $f \in \mathcal{A} \cap RV_\rho$, $\rho > 0$. Then there is a continuous and strictly increasing function $f_1 \in \mathcal{A} \cap RV_\rho$, $\rho > 0$, such that $f_1(x) \sim f(x)$, as $x \to +\infty$ (see, e.g. [3]).

Contrary to the statement, assume that $g^{\leftarrow} \in \langle f^{\leftarrow} \rangle$ whenever $g \in \{f\} \cap \mathcal{A}$. Since $f^{\leftarrow} \in RV_{1/\rho}$ we have that $g^{\leftarrow} \in RV_{1/\rho}$ and so $g \in RV_\rho$.

We find that $g \in \mathcal{A}$ and $1 \leq \dfrac{f_1(x)}{g(x)} < 2$ for all $x \geq a$, so $g \in \{f\}$. However, $g \notin RV$ because $\dfrac{g(x+1)}{g(x)} \nrightarrow 1$ for $x \to +\infty$. Namely, consider the sequence (α_n) defined by $\alpha_n = a_{n+1} - \min\{1/2, a_{n+1} - a_n\}$, $n \in \mathbb{N}$. Since $\alpha_n \to +\infty$, as $n \to +\infty$ and $\dfrac{g(\alpha_n + 1)}{g(\alpha_n)} \geq 2$, for every $n \in \mathbb{N}$, we find that $\overline{\lim_{n\to+\infty}} \dfrac{g(x+1)}{g(x)} \geq 2$. Since $g \in RV \subsetneq IRV$ we have $\lim_{x\to+\infty} \dfrac{g(x+1)}{g(x)} = 1$, which contradicts to $g \in RV_\rho$. □

Remark 2.7. The characterization of f obtained in Proposition 2.6 is true for every $g \in \{f\}$.

Proposition 2.8. *Let $f \in \mathcal{A}$.*

(a) If $f \in PI$, then $g^{\leftarrow} \in \{f^{\leftarrow}\}$ for every $g \in \langle f \rangle \cap \mathcal{A}$.

(b) If $g^{\leftarrow} \in \{f^{\leftarrow}\}$ whenever $g \in \langle f \rangle \cap \mathcal{A}$, then $f \in PI$. The same conclusion holds for every $g \in \langle f \rangle$.

Proof. (a) Obviously, $f^{\leftarrow} \in ORV$ and $g \in \langle f \rangle$. Then $g \in \{f\}$, and according to [9] we find that $g^{\leftarrow} \in \{f^{\leftarrow}\}$.

(b) For an arbitrary fixed $\alpha > 0$ define $g(x) = \alpha \cdot f(x)$, $x \geq a$. Then $g \in \mathcal{A} \cap \langle f \rangle$, $g^{\leftarrow}(x) = f^{\leftarrow}(\dfrac{1}{\alpha}x)$ for all sufficiently large x, and

$$0 < m(\alpha) \leq \underline{\lim_{x\to+\infty}} \frac{g^{\leftarrow}(x)}{f^{\leftarrow}(x)} = \underline{\lim_{x\to+\infty}} \frac{f^{\leftarrow}(\frac{1}{\alpha})}{f^{\leftarrow}(x)} \leq \overline{\lim_{x\to+\infty}} \frac{g^{\leftarrow}(x)}{f^{\leftarrow}(x)} = \overline{\lim_{x\to+\infty}} \frac{f^{\leftarrow}(\frac{1}{\alpha}x)}{f^{\leftarrow}(x)} \leq M(\alpha) < +\infty.$$

Since $f^{\leftarrow} \in ORV$, by [9] we finally get $f \in PI$. □

Corollary 2.9. *Let $f \in \mathcal{A}$.*

(a) If $f \in ORV$, then $g \in \{f\}$ whenever $g \in \mathcal{A}$ and $g^{\leftarrow} \in \langle f^{\leftarrow} \rangle$.

(b) If $g \in \{f\}$, whenever $g \in \mathcal{A}$ and $g^{\leftarrow} \in \langle f^{\leftarrow} \rangle$, then $f \in ORV$. The same conclusion holds for every $g \in \{f\}$.

Proof. (a) Since $f^{\leftarrow} \in PI$, by Proposition 2.8 (a) we have that $(g^{\leftarrow}(x))^{\leftarrow} \asymp (f^{\leftarrow}(x))^{\leftarrow}$, as $x \to +\infty$. Since $f \in ORV$ and for every $\beta > 1$ we have $1 \leq \dfrac{(f^{\leftarrow}(x))^{\leftarrow}}{f(x)} \leq \dfrac{f(\beta x)}{f(x)}$ for all sufficiently large x, it follows

$$1 \leq \underline{\lim}_{x\to+\infty} \frac{(f^{\leftarrow}(x))^{\leftarrow}}{f(x)} \leq \overline{\lim}_{x\to+\infty} \frac{(f^{\leftarrow}(x))^{\leftarrow}}{f(x)} \leq \overline{\lim}_{x\to+\infty} \frac{f(\beta x)}{f(x)} = \overline{k}_f(\beta) < +\infty.$$

Hence $(f^{\leftarrow}(x))^{\leftarrow} \asymp f(x)$ as $x \to +\infty$. Next, since $g^{\leftarrow} \in \langle f^{\leftarrow} \rangle$, there is an $\alpha > 0$ such that $g^{\leftarrow}(x) = h(x) \cdot f^{\leftarrow}(x)$ for all sufficiently large x, where $h(x) \to \alpha$ as $x \to +\infty$. Accordingly, for $\gamma > \gamma_0 \geq 1$ we have

$$\underline{\lim}_{x\to+\infty} \frac{g^{\leftarrow}(\gamma x)}{g^{\leftarrow}(x)} \geq \underline{\lim}_{x\to+\infty} \frac{f^{\leftarrow}(\gamma x)}{f^{\leftarrow}(x)} > 1,$$

and hence $g^{\leftarrow} \in PI$, i.e. $g \in ORV$. Similarly as in previous part of the proof we find that $(g^{\leftarrow}(x))^{\leftarrow} \asymp g(x)$ as $x \to +\infty$. Therefore $f(x) \asymp g(x)$ as $x \to +\infty$.

(b) Let $g^{\leftarrow}(x) = \alpha \cdot f^{\leftarrow}(x)$ for $x \geq a$ and some $\alpha > 0$. Further, define $h^{\Leftarrow}(x) = (h^{\leftarrow}(x))^{\leftarrow}$ for $h \in \mathcal{A}$ and all sufficiently large x. Then we have

$$(g^{\leftarrow}(x))^{\leftarrow} = f^{\Leftarrow}\Big(\frac{1}{\alpha}x\Big)$$

for all sufficiently large x, so

$$\overline{\lim}_{x\to+\infty} \frac{f^{\Leftarrow}(\frac{1}{\alpha}x)}{f^{\Leftarrow}(x)} = \overline{\lim}_{x\to+\infty} \frac{g^{\Leftarrow}(x)}{f^{\Leftarrow}(x)} \leq M(\alpha) < +\infty.$$

Therefore, $(f^{\leftarrow}(x))^{\leftarrow} \in ORV$, and consequently $f \in ORV$, because $(f^{\leftarrow}(x))^{\leftarrow} \asymp f(x)$, as $x \to +\infty$. Similarly, $g \in ORV$ for every $g \in \{f\}$. □

Corollary 2.10. *Let $f \in \mathcal{A}$.*

(a) If $f \in SV$, then $g \in [f]_{\sim} \subsetneq \langle f \rangle$ whenever $g \in \mathcal{A}$ and $g^{\leftarrow} \in \{f^{\leftarrow}\}$.

(b) If $g \in \langle f \rangle$ whenever $g \in \mathcal{A}$ and $g^{\leftarrow} \in \{f^{\leftarrow}\}$, then $f \in SV \cup RV_\rho$, $\rho > 0$.

(c) For every $f \in RV_\rho$, $\rho > 0$, there is a $g \in \mathcal{A} \setminus \langle f \rangle$ such that $g^{\leftarrow} \in \{f^{\leftarrow}\}$.

Proof. (a) If $f \in SV$, then $f^{\leftarrow} \in R_\infty$ and by [10] it follows that $(g^{\leftarrow}(x))^{\leftarrow} \sim (f^{\leftarrow}(x))^{\leftarrow}$ as $x \to +\infty$. Since $f \in SV$, we have $(f^{\leftarrow}(x))^{\leftarrow} \sim f(x)$ for $x \to +\infty$, accordingly to [12]. Similarly, we have $g^{\leftarrow} \in R_\infty$, and hence by [13] we obtain $g \in SV$ and $(g^{\leftarrow}(x))^{\leftarrow} \sim g(x)$ when $x \to +\infty$. This means that $g \in [f] \subsetneq \langle f \rangle$.

(b) Since $g \in \langle f \rangle$ whenever $g^{\leftarrow} \in \{f^{\leftarrow}\}$, so $g \in \langle f \rangle$ whenever $g^{\leftarrow} \in \langle f^{\leftarrow} \rangle$. By Proposition 2.4 it follows that $f \in SV \cup RV_\rho$, $\rho > 0$. The same conclusion holds for every $g \in \langle f \rangle$.

(c) Finally, let $f \in \mathcal{A} \cap RV_\rho$, $\rho > 0$. Then there is a continuous and strictly increasing function $F \in \mathcal{A} \cap RV_{1/\rho}$, $\rho > 0$, such that $F(x) \sim f^{\leftarrow}(x)$, as $x \to +\infty$.

Contrarily to the statement, assume that $g \in \langle f \rangle$ whenever $g \in \mathcal{A}$ and $g^{\leftarrow} \in \{f^{\leftarrow}\}$. Since $f \in RV_\rho$ we have that $g \in RV_\rho$, so $g^{\leftarrow} \in RV_{1/\rho}$.

Next, define the sequence $a_1 < a_2 < \cdots$ by $a_1 = a$ and $a_n = F^{-1}\Big(2 \cdot F(a_{n-1})\Big)$ $(n \geq 2)$. Then $g^{\leftarrow}(x) = F(a_n)$ $(a_n \leq x < a_{n+1})$ $(n \in N)$.

Hence we find $1 \leq \dfrac{F(x)}{g^{\leftarrow}(x)} < 2$ for all $x \geq a$, so $g^{\leftarrow} \in \{f^{\leftarrow}\}$ and $g^{\leftarrow} \notin RV$, since $\dfrac{g^{\leftarrow}(x+1)}{g^{\leftarrow}(x)} \nrightarrow 1$ when $x \to +\infty$. But this, similarly as in the proof of Proposition 2.6, contradicts to $g^{\leftarrow} \in RV_{1/\rho}$. □

Remark 2.11. If in the proof of Corollary 2.10 (b) we take: $g(x) = a_1$ $(F(a_1)/2 \leq x < F(a_1))$ and $g(x) = a_n$ $(F(a_n) \leq x < F(a_{n+1}))$ $(n \geq 2, x \geq a)$, then $g \in \mathcal{A}$.

Remark 2.12. The characterization of f obtained in Corollary 2.10 is true for every $g \in \langle f \rangle$.

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