Filomat 29:4 (2015), 781–785
DOI 10.2298/FIL1504781E

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Summation of Double Sequences and Selection Principles

Nebojša Elez[a], Dragan Djurčić[b], Aleksandar Šebeković[b]

[a] *University of East Sarajevo, Faculty of Philosophy, Alekse Šantića 1, 71420 Pale, Bosnia and Hercegovina*
[b] *University in Kragujevac, Faculty of Technical Sciences Čačak, Svetog Save 65, 32000 Čačak, Serbia*

Abstract. This paper proves some selection properties of a class of positive real double sequences which converge to 0 in the sense of Pringsheim (see, e.g. [10]).

1. Introduction

For a real double sequence $x = (x_{m,n})$ we say that it is convergent to 0 in the sense of Pringsheim (denoted by $P\text{-}\lim x = 0$) [10] if

$$\lim_{\min\{m,n\}\to+\infty} x_{m,n} = 0. \tag{1}$$

Denote the class of such double sequences by c_2^0 and the class of such positive double sequences by $c_{2,+}^0$ (see [2]).

Theory of double sequences (in particular, theory of convergence of double sequences in Pringsheim's sense) is important current part of mathematical analysis and other mathematical disciplines (see, e.g. [6, 7, 9, 11]).

Let $x = (x_{m,n})$ be a double sequence of real numbers. Then:

1° $\omega^{(d)}(x) = (\omega_n^{(d)}(x))$ is the Landau-Hurwicz sequence of x, if

$$\omega_n^{(d)}(x) = \sup\{|x_{k,l} - x_{r,s}| \mid k \geq n,\ l \geq n,\ r \geq n \text{ and } s \geq n\}$$

for all $n \in \mathbb{N}$ (see [1]);

2° $S_1^*(x) = (S_n(x))$, where $S_n(x) = \sum\limits_{k=1}^{n}\left(\sum\limits_{l=1}^{n} x_{k,l}\right)$ for $n \in \mathbb{N}$, is the sequence which represents the diagonal series of x;

3° x is with finite diagonal sum, if there exists $S_1^{(x)} \in \mathbb{R}$ such that $S_1^{(x)} = \lim\limits_{n\to+\infty} S_n(x)$ (denoted by $S_1^{(x)} = \sum x$);

2010 *Mathematics Subject Classification*. Primary 40A05, 54A20
Keywords. Pringsheim convergence, selection principles, translationally rapidly varying sequence.
Received: 16 October 2013; Accepted: 01 December 2013
Communicated by Dragan S. Djordjević
This paper is supported by the Ministry of Education, Science and Technological Development of Republic of Serbia, Grant No. 174026.
Email addresses: jasnaelez@gmail.com (Nebojša Elez), dragan.djurcic@tfc.kg.ac.rs (Dragan Djurčić), aleksandar.sebekovic@ftn.kg.ac.rs (Aleksandar Šebeković)

4° $S_2^*(x) = (S_{m,n}(x))$, where $S_{m,n}(x) = \sum_{k=1}^{m}\left(\sum_{l=1}^{n} x_{k,l}\right)$ for $m \in \mathbb{N}$ and $n \in \mathbb{N}$, is the double sequence which represents the double series of x;

5° x is with finite sum in the Pringsheim sense, if there exists $S_2^{(x)} \in \mathbb{R}$ such that $S_2^{(x)} = P\text{-}\lim S_2^*(x)$ (denoted by $S_2^{(x)} = P\text{-}\sum x$);

Let us take a double sequence $x = (x_{m,n})$ of positive real numbers (denoted by $x \in \mathbb{S}_2$). Then:

6° x is an element of a class l_2^1 if $S_1^{(x)} \in \mathbb{R}$ and x is an element of a class $P\text{-}l_2^1$ if $S_2^{(x)} \in \mathbb{R}$;

7° x is an element of a class $l^1_{2,\mathsf{Tr}(R_{-\infty,S})}$ if $x \in l_2^1$ and $\omega^{(d)}(S_2^*(x)) \in \mathsf{Tr}(R_{-\infty,S})$.

Remark 1.1. *(a) For a sequence $a = (a_n)$ of positive numbers we say that it is translationally rapidly varying in the sense of de Haan with the index of variability $-\infty$, if*

$$\lim_{n\to+\infty} \frac{a_{[n+\alpha]}}{a_n} = 0 \tag{2}$$

for every $\alpha \geq 1$ (the class of such sequences is denoted by $\mathsf{Tr}(R_{-\infty,S})$). Such sequences are important objects in asymptotic analysis (see, e.g. [3]). It holds that the class $\mathsf{Tr}(R_{-\infty,S})$ is a proper subclass of the class of positive real sequences which converge to 0.

(b) For a double sequence x of positive real numbers it holds that x converges in Pringsheim's sense in $\mathbb{R}$ if and only if $\omega_n^{(d)}(x) \to 0$, for $n \to +\infty$ (see [1]).

(c) For a sequence $b = (b_n)$ of real numbers the sequence $\omega(b) = (\omega_n(b))$ is the Landau-Hurwicz sequence, if $\omega_n(b) = \sup\{|b_k - b_r| \mid k \geq n \text{ and } r \geq n\}$ for every $n \in \mathbb{N}$ (see [4]). The fact $\omega^{(d)}(S_2^*(x)) \in \mathsf{Tr}(R_{-\infty,s})$ from 6° is equal to the fact that $\omega(S_1^*(x)) \in \mathsf{Tr}(R_{-\infty,s})$.

Proposition 1.2. *Let $x = (x_{m,n}) \in \mathbb{S}_2$. For $S \in (0,+\infty)$ it holds that $S = P\text{-}\sum x$ if and only if $S = \sum x$.*

Proof.

(⇒) Let $S \in (0,+\infty)$ and $S = P\text{-}\sum x$. Then for $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $S - S_{m,n}(x) \leq \varepsilon$ for every $m \geq n_0$ and every $n \geq n_0$. So, for $n \geq n_0$ we have $S - S_{n,n}(x) \leq \varepsilon$, so $S = \sum x$.

(⇐) Let $S \in (0,+\infty)$ and $S = \sum x$. Then for every $\varepsilon > 0$ there is $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $S - S_{n,n}(\varepsilon)(x) \leq \varepsilon$ for every $n \geq n_0$. For $k, l \in \{0\} \cup \mathbb{N}$ it holds

$$\begin{aligned} 0 &< S - S_{n_0+\max\{l,k\},n_0+\max\{l,k\}}(x) \leq S - S_{n_0+k,n_0+l}(x) \\ &\leq S - S_{n_0+\min\{l,k\},n_0+\min\{l,k\}}(x) \leq \varepsilon, \end{aligned}$$

so $S = P\text{-}\sum x$. □

Proposition 1.3. *Let $x = (x_{m,n}) \in \mathbb{S}_2$. If $\sum x$ is an element of $\mathbb{R}$ ($x \in l_2^1$), then $P\text{-}\lim x = 0$. The converse need not be true.*

Proof. If $x \in \mathbb{S}_2$ and $x \in l_2^1$, then for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that

$$V_n(x) = x_{1,n} + x_{2,n} + \cdots + x_{n,n} + x_{n,n-1} + \cdots + x_{n,2} + x_{n,1} \leq \varepsilon$$

for every $n \geq n_0$. So, for every $k, l \in \{0\} \cup \mathbb{N}$ it holds that $x_{n_0+k,n_0+l} \leq \varepsilon$, i.e. $x \in c^0_{2,+}$.

Now, let us observe that the double sequence $x = (x_{m,n})$, where $x_{m,n} = \frac{1}{\max\{m,n\}}$ for $m \in \mathbb{N}$ and $n \in \mathbb{N}$. Then $\sum x > \sum_{n=1}^{+\infty} \frac{1}{n} = +\infty$, so the double sequence x does not have finite diagonal sum. On the other side, $x \in c^0_{2,+}$, because for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ (let say $n_0 = [\frac{1}{\varepsilon}] + 1$), such that $x_{m,n} \leq \varepsilon$ for every $m \geq n_0$ and every $n \geq n_0$. □

Corollary 1.4. *Let $x = (x_{m,n}) \in \mathbb{S}_2$. If P-$\sum x$ is an element of $\mathbb{R}$ ($x \in P$-l_2^1), then P-$\lim x = 0$. The converse need not be true.*

Let $\mathcal{A}$ and $\mathcal{B}$ be non-empty subsets from $\mathbb{S}_2$. Let us define the following selection principles (see, e.g. [5] and [1]):

(a) $S_1^{(d)}(\mathcal{A}, \mathcal{B})$ denotes the selection principle: for every double sequence $(A_{m,n})$ of elements from $\mathcal{A}$ there exists an element B from $\mathcal{B}$ such that $B = (b_{m,n})$ and $b_{m,n} \in A_{m,n}$ for every $m, n \in \mathbb{N}$;

(b) $\alpha^{(d)}(\mathcal{A}, \mathcal{B})$ denotes the selection principle: for every double sequence $(A_{m,n})$ of elements from $\mathcal{A}$ there exists an element $B \in \mathcal{B}$ such that the set $B \cap A_{m,n}$ is infinite for every $m, n \in \mathbb{N}$;

(c) $S_1^{\varphi}(\mathcal{A}, \mathcal{B})$ denotes the selection principle: for every sequence (A_t) of elements from $\mathcal{A}$ there exists an element B from $\mathcal{B}$ such that $B = (b_{m,n})$ and $b_{m,n} \in A_t$ for $t = \varphi(m, n)$, where $\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is an in advance given bijection.

2. Main Results

The following propositions improve results given in [1] and [5].

Proposition 2.1. *The selection principle $S_1^{(d)}(c_{2,+}^0, l^1_{2,\mathrm{Tr}(R_{-\infty,S})})$ is satisfied.*

Proof. Let a double sequence of double sequences $(x_{m,n,k,l})$ be given, where for every $(k_0, l_0) \in \mathbb{N} \times \mathbb{N}$ it holds $x^{(k_0,l_0)} = (x_{m,n,k_0,l_0}) \in c_{2,+}^0$. Let us create the double sequence $y = (y_{k,l})$ in the following way:

(step 1) take $y_{1,1}$ from the double sequence $x^{(1,1)}$ so that $y_{1,1} \leq 1$;

(step 2) for $(k, l) \in \{(1, 2), (2, 2), (2, 1)\}$ take $y_{k,l}$ from the double sequence $x^{(k,l)}$ so that $y_{k,l} \leq \frac{1}{2^2} \cdot \frac{V_1(y)}{2 \cdot 2 - 1}$;

(step n, $n \geq 3$) for $(k, l) \in \{(1, n), (2, n), \ldots, (n, n), \ldots, (n, 2), (n, 1)\}$ take $y_{k,l}$ from the double sequence $x^{(k,l)}$ such that $y_{k,l} \leq \frac{1}{n^2} \cdot \frac{V_{n-1}(y)}{2n-1}$. We have that y is a double sequence of positive real numbers and that for every $n \in \mathbb{N}$ it holds $S_{n,n}(y) \leq 1 + \cdots + \frac{1}{n^2}$, because $S_{n,n}(y) = \sum_{p=1}^{n} V_p(y)$. So, there exists $S_1^{(y)} > 0$ such that $S_1^{(y)} = \lim_{n\to+\infty} S_{n,n}(y)$, so $y \in l_2^1$. Let us observe now that $\omega^{(d)}(S_2^*(y)) = (\omega_n^{(d)}(S_2^*(y)))$. Then for every $n \in \mathbb{N}$ it holds $\omega_n^{(d)}(S_2^*(y)) = S_1^{(y)} - S_n(y)$. Also, for sufficiently large $n \in \mathbb{N}$ the following holds:

$$
\begin{aligned}
\frac{\omega_{n+1}^{(d)}(S_2^*(y))}{\omega_n^{(d)}(S_2^*(y))} &= \frac{S_1^{(y)} - S_{n+1}(y)}{S_1^{(y)} - S_n(y)} = 1 - \frac{S_{n+1}(y) - S_n(y)}{S_1^{(y)} - S_n(y)} = \\
&= 1 - \frac{V_{n+1}(y)}{V_{n+1}(y) + V_{n+2}(y) + \cdots} = 1 - \frac{1}{1 + \frac{V_{n+2}(y)}{V_{n+1}(y)} + \frac{V_{n+3}(y)}{V_{n+1}(y)} + \cdots} = \\
&= 1 - \frac{1}{1 + \frac{V_{n+2}(y)}{V_{n+1}(y)} + \frac{V_{n+3}(y)}{V_{n+2}(y)} \cdot \frac{V_{n+2}(y)}{V_{n+1}(y)} + \cdots} \leq \\
&\leq 1 - \frac{1}{1 + \frac{V_{n+2}(y)}{V_{n+1}(y)} + \frac{V_{n+3}(y)}{V_{n+2}(y)} + \cdots}.
\end{aligned}
$$

Here we used the fact that the series $\sum_{n=1}^{+\infty} \frac{V_{n+1}(y)}{V_n(y)}$ is convergent and because of that $\sum_{k=n}^{+\infty} \frac{V_{k+1}(y)}{V_k(y)} \to 0$ for $n \to +\infty$. Thus, the following holds:

$$
\lim_{n\to+\infty} \frac{\omega_{n+1}^{(d)}(S_2^*(y))}{\omega_n^{(d)}(S_2^*(y))} = 0,
$$

so $\omega^{(d)}(S_2^*(y)) \in \mathrm{Tr}(R_{-\infty,S})$. Thus, $y \in l^1_{2,\mathrm{Tr}(R_{-\infty,S})}$. From Proposition 1.3 it follows $y \in c^0_{2,+}$ and from the construction of the double sequence y it follows that $y_{k,l} \in x^{(k,l)}$ for every $(k,l) \in \mathbb{N}\times\mathbb{N}$. This ends the proof. □

Corollary 2.2. *The selection principle $S_1^{(d)}(c^0_{2,+}, l^1_2)$ is satisfied.*

Remark 2.3. *From Proposition 1.2 and Corollary 2.2 it follows that the selection principle $S_1^{(d)}(c^0_{2,+}, P\text{-}l^1_2)$ is satisfied.*

Proposition 2.4. *The selection principle $\alpha_2^{(d)}(c^0_{2,+}, l^1_2)$ is satisfied.*

Proof. Let $(x_{m,n,k,l})$ be a double sequence of double sequences with properties as in the proof of Proposition 2.1. Let us sort it (applying some of standard methods) into a sequence of double sequences $(x_{m,n,r})$, where for every $r_0 \in \mathbb{N}$ it is fulfilled $x^{(r_0)} = (x_{m,n,r_0}) \in c^0_{2,+}$. Let form the double sequence $y = (y_{s,t})$ in the following way:

(step 1) let $y_0 = (y^{(0)}_{s,t})$ be a double sequence such that for every $(s,t) \in \mathbb{N}\times\mathbb{N}$ it holds $0 < y^{(0)}_{s,t} \leq \frac{1}{M^2(2M-1)}$, where $M = \max\{s,t\}$;

(step 2) let (p_r) be a sequence of prime numbers in ascending order, where $p_1 = 2$. For $r \in \mathbb{N}$ form the double sequence $y_r = (y^{(r)}_{s,t})$ by making changes in the double sequence y_{r-1} at positions $(p_r^{\varphi_r(g)}, p_r^{\varphi_r(g)})$, $g \in \mathbb{N}$ (where $\varphi_r : \mathbb{N} \to \mathbb{N}$ is a bijection such that the sequence $\sum\limits_{g=1}^{+\infty} x_{p_r^{\varphi_r(g)}, p_r^{\varphi_r(g)}, r}$ is convergent and $\leq \frac{1}{r^2}$), in the following way: replace elements in y_{r-1} at the mentioned positions with elements from the double sequence $x^{(r)}$ at the same positions, respectively.

Let $y = \lim\limits_{r\to+\infty} y_r$. Then the double sequence $y = (y_{s,t}) \in c^0_{2,+}$ and $y \in l^1_2$. According to the construction of y there are infinitely many common elements of y with every double sequence $x^{(r)}$, $r \in \mathbb{N}$, at the same positions, which ends this proof. □

Remark 2.5.

(a) *According to Propositions 1.2 and 2.4 the selection principle $\alpha_2^{(d)}(c^0_{2,+}, P\text{-}l^1_2)$ is satisfied.*

(b) *Also, selection principles $\alpha_j^{(d)}(c^0_{2,+}, \mathcal{B})$ are satisfied for $\mathcal{B} \in \{l^1_2, P\text{-}l^1_2\}$ and $i \in \{3,4\}$, as well as selection principles $\alpha_j(c^0_{2,+}, \mathcal{B})$ for $j \in \{2,3,4\}$ (about these selection principles see [1] and [8]).*

Proposition 2.6. *The selection principle $S_1^{\varphi}(c^0_{2,+}, l^1_2)$ is satisfied.*

Proof. Let $\varphi : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ be a bijection and let a sequence of double sequences $(x_{m,n,r})$ be given, where for every $r_0 \in \mathbb{N}$, $x^{(r_0)} = (x_{m,n,r_0}) \in c^0_{2,+}$ is fulfilled. Create the double sequence $y = (y_{s,t})$ in the following way:

Let arbitrary $r \in \mathbb{N}$ be fixed. Also, let $(s(r), t(r)) = \varphi^{-1}(r)$ and let $M(r) = \max\{s(r), t(r)\}$. There exists $n_0(r) \in \mathbb{N}$ such that $x_{n_0(r),n_0(r),r} \leq \frac{1}{M^2(r)(2M(r)-1)}$. Take $y_{s(r),t(r)} = x_{n_0(r),n_0(r),r}$, for $r \in \mathbb{N}$, and in that way create the double sequence $(y_{s(r),t(r)})$. It follows that $y \in c^0_{2,+}$ and $y \in l^1_2$. According to the construction of double sequence y it follows that y and $x^{(r)}$ have exactly one common element for each $r \in \mathbb{N}$. This ends the proof. □

Remark 2.7. *According to Propositions 1.2 and 2.6 the selection principle $S_1^{\varphi}(c^0_{2,+}, P\text{-}l^1_2)$ holds.*

References

[1] A. Caserta, D. Djurčić, M. Mitrović, *Selection principles and double sequences* II, Hacettepe J. Math. Stat., to appear.

[2] D. Djurčić, M.R. Žižović, A. Petojević, *Note on selection principles of Kočinac*, Filomat 26 (2012), 1291–1295.

[3] D. Djurčić, Lj.D.R. Kočinac, M.R. Žižović, *Classes of sequences of real numbers, games and selection properties*, Topology Appl. 156 (2008), 46–55.

[4] D. Djurčić, Lj.D.R. Kočinac, M.R. Žižović, *On the class* $\mathbb{S}_{\underline{0}}$ *of real sequences*, Appl. Math. Letters 25 (2012), 1296–1298.

[5] D. Djurčić, Lj.D.R. Kočinac, M.R. Žižović, *Double sequences and selections*, Abst. Appl. Anal. 2012 (2012), Article ID 497594, 6 pages.

[6] E.W. Hobson, The theory of functions of a real variable, Vol. 2, Cambridge University Press, Cambridge, UK, 2nd. edition, 1926.

[7] K. Kayduman, C. Çakan, *The Cesaro core of double sequences*, Abst. Appl. Anal. 2011 (2011), Article ID 950364, 9 pages.

[8] Lj.D.R. Kočinac, *On the* α_i*-selection principles and games*, Contem. Math. 533 (2011), 107–124.

[9] R.F. Patterson, E. Savaş, *A cathegory theorem for double sequences*, Appl. Math. Letters 24 (2011), 1680–1684.

[10] A. Pringsheim, *Zur Theorie der zweifach unendlichen Zahlenfolgen*, Math. Ann. 53 (1900), 289–321.

[11] E. Savaş, (A_σ)*-double sequence spaces defined by Orlicz function and double statistical convergence*, Comput. & Math. Appl. 55 (2008), 1293–1301.