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Semigroups with apartness

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Proving a constructive version of the Spectral Mapping Theorem, Bridges and Havea used a constructive semigroup with inequality in [8]. This motivated us to achieve a little progress in that direction. The starting point is the structure $(S, =, \neq, \cdot)$ called a semigroup with apartness. Our primary objective is to prove isomorphism theorems for such constructive semigroups. In doing so our main ideas and notions come from [10].

1 Introduction

In this paper, we present a small portion of the theory of semigroups with apartness from a constructive-algebraic point of view. The focus is on Bishop's approach to constructive mathematics (**BISH**). Since the appearance of Bishop's monograph [2] in 1967, there have been significant developments in Bishop-style analysis and topology; cf., e.g., [4, 8, 9, 10, 23]. But, as is remarked in [7], "contrary to Bishop's expectations, modern algebra also proved amenable to natural, thoroughgoing, constructive treatment"—cf. the well-known constructive algebra book [20].

In [24], Troelstra and van Dalen say that "[t]he study of algebraic structures in an intuitionistic setting was undertaken by Heyting, [15]. Heyting considered structures equipped with an apartness relation in full generality". Since then several authors have worked in this area; cf., e.g., [20, 23, 24]. There is no doubt about the deep connections of constructive analysis with computer science: cf., e.g., [7]. In [19] it is shown that constructive algebraic structures with apartness also can be applied in computer science (especially in computer programming).

In universal algebra within **CLASS** (i.e., classical mathematics), the formulation of homomorphic images (together with subalgebras and direct products) is one of the principal tools used to manipulate algebras. In the study of homomorphic images of an algebra a lot of help comes from the notion of a quotient algebra, which captures all homomorphic images, at least up to isomorphism. On the other hand, homomorphism is the concept which goes hand in hand with congruences. Thus concepts of congruence, quotient algebra and homomorphism are closely related. Knowing that the congruence θ on an algebra A is the kernel of the quotient map from A onto A/θ, we can treat congruence relations on A as kernels of homomorphisms with A as the domain. The relationship between quotients, homomorphisms and congruences is described by the celebrated *isomorphism theorems*, which are a general and important foundational part of universal algebra; cf. [11].

The main goal of this paper is to give isomorphism theorems for semigroups with apartness. As in [4, 8], "every effort will be made to follow classical development along the lines suggested by familiar classical theories or in altogether new directions."

Semigroup theory is also a young part of **CLASS**. As a separate branch of algebra with its own objects, formulations of problems, and methods of investigations, semigroup theory was formed more than 60 years ago. Historically, it can be viewed as an algebraic abstraction of the properties of the composition of transformations on a set. Later sources, besides of those coming from the theory of groups and the theory of rings, include an abstraction of certain ideas arising in connection with topological or linear spaces. More about the history of

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algebraic theory of semigroups can be found in [17]. Within **BISH**, the history of constructive semigroups with an inequality began recently. For example, in proving a constructive version of the Spectral Mapping Theorem, Bridges and Havea [6], used such semigroups.

Within **BISH** isomorphism theorems for groups and rings with *tight* apartness are presented in [23, 24]. Beside the fact that "a considerable part of the traditional notions and techniques is salvaged", new concept must be introduced as well. Thus, as a "positive 'dual' of" a subgroup the notion of antisubgroup appears in groups with tight apartness. A similar job within rings with tight apartness is done by the notion of anti-ideal.

As we have already pointed out, our primary objective is to prove isomorphism theorems for semigroups with apartness. Our work is based on applications of ideas and notions coming from [6, 10]. The paper is organized in the following way. A set with apartness together with an equivalence and its 'dual' coequivalence (Theorem 2.3) is the subject of § 2. The main result of this section is Theorem 2.5, an apartness version of the Isomorphism Theorem for Sets. Semigroups with apartness are studied in Section 3. This section begins with Example 3.1, which justify the study of such structures. The main results are given in Theorem 3.3 and Theorem 3.4 which are *apartness isomorphism theorems* for semigroups with apartness.

For undefined notions and notation, cf. [20, 23, 24]. Other general references for constructive mathematics are [2, 4, 8, 9, 10]. For the classical case, cf. [1, 11, 18, 21].

2 Sets with apartness

In the constructive order theory the notion of *cotransitivity*, i.e., the property that for every pair of related elements any other element is related to one of the original elements in the same order as the original pair is a constructive counterpart to classical transitivity. Irreflexive and cotransitive relations are the building blocks of constructive order theory. With a primitive notion of 'set with apartness', our intention is to connect all relations defined on such a set. This is done by requiring them to be a part (subset) of an apartness. Such relations are clearly irreflexive; if, in addition, they are cotransitive, then they are called *coquasiorders*. More about coquasiorders can be found in, e.g., [13, 22]. The main subjects of this section are those symmetric coquasiorders called *coequivalences*, and the equivalences which can be associated to them.

Following [3] and [16], we define a set S by giving an algorithm for constructing members of S, together with a prescribed equivalence relation $=$, called the *equality* of S. Furthermore, we are interested only in properties $P(x)$ which are *extensional* in the sense that for all $x, x' \in S$ with $x = x'$, $P(x)$ and $P(x')$ are equivalent. Let $(S, =)$ be an *inhabited* set—that is, one in which we can construct an element. By an *apartness* on S (cf. [20]), we mean a binary relation $\neq$ on S which satisfies the axioms of irreflexivity, symmetry and cotransitivity:

$$\neg(x \neq x) \tag{Ap1}$$

$$x \neq y \;\Rightarrow\; y \neq x, \tag{Ap2}$$

$$x \neq z \;\Rightarrow\; \forall y\,(x \neq y \;\vee\; y \neq z). \tag{Ap3}$$

We then say that $(S, =, \neq)$—or just S itself—is a *set with apartness*. The apartness on a set S is *tight* if

$$\neg(x \neq y) \;\Rightarrow\; x = y. \tag{Ap4}$$

The term "pre apartness" for an apartness relation is used in [23, 24]. By extensionality we have

$$x \neq y \wedge y = z \;\Rightarrow\; x \neq z. \tag{Ap5}$$

A set with apartness $(S, =, \neq)$ is the starting point for our further considerations, and will be simply denoted by S.

Let $f : S \longrightarrow T$ be a mapping (function) of sets with apartness. The well-definedness of f, i.e., $\forall x, y \in S\,(x =_S y \;\Rightarrow\; f(x) =_T f(y))$, follows by extensionality. Two mappings $f : S \longrightarrow T$, and $g : S \longrightarrow T$ are *equal* if, as usual, $\forall x \in S\,(f(x) =_T g(x))$. The set of all mappings from a set S to a set T will be denoted by T^S. We can define relation $\neq$ on T^S in the following way $f \neq g \;\Leftrightarrow\; \exists x \in S\,(f(x) \neq_T g(x))$. The following result taken from [20], will be of help.

Theorem 2.1 *The structure $(T^S, =, \neq)$ is a set with apartness.*

A mapping $f : S \longrightarrow T$ is *onto* T if $\forall y \in T\ \exists x \in S\ (y =_T f(x))$; it is *one-one* if $f(x) =_T f(y) \Rightarrow x =_S y$; it is a *strongly extensional* mapping, or, for short, *se-mapping*, if $\forall x \in S\ \forall y \in S\ (f(x) \neq_T f(y) \Rightarrow x \neq_S y)$; it is *injective* if $x \neq_S y \Rightarrow f(x) \neq_T f(y)$; it is a *bijection* between S and T if it is a one-one mapping from S onto T; and it is an *apartness bijection* between S and T if it is injective se-bijection.

Let Y be a subset of S. Following [5, 6,10], we define the subset

$$\sim Y = \{x \in S : x \rhd Y\}$$

of S called the *complement of Y in S*, where $\rhd$, a relation between an element $x \in S$ and the subset Y, is defined by

$$x \rhd Y \Leftrightarrow \forall y \in Y\ (x \neq y).$$

The *Cartesian product* of two sets with apartness $(S, =_S, \neq_S)$ and $(T, =_T, \neq_T)$ is the set of all ordered pairs $(s, t) \in S \times T$ with $s \in S$ and $t \in T$, together with the equality and the apartness given by

$$(s, t) = (u, v) \quad \Leftrightarrow \quad s =_S u \wedge t =_T v,$$

$$(s, t) \neq (u, v) \quad \Leftrightarrow \quad s \neq_S u \vee t \neq_T v.$$

Let $(S \times S, =, \neq)$ be a set with apartness. Then for any nonempty subset $\tau \subseteq S \times S$ (i.e., for any relation τ defined on the set S) and any $(a, b) \in S \times S$ we have, by the definition given above,

$$(a, b) \rhd \tau \quad \Leftrightarrow \quad \forall (x, y) \in \tau\ ((a, b) \neq (x, y)).$$

Among relations defined on S we consider only those which are related to the apartness in the following way:

$$\alpha \subseteq \neq. \qquad \text{(Q1)}$$

We call such a relation α *consistent*. A relation α is *cotransitive* if

$$(x, z) \in \alpha \implies \forall y\ ((x, y) \in \alpha \vee (y, z) \in \alpha). \qquad \text{(Q2)}$$

A consistent, cotransitive relation τ defined on S is called a *coquasiorder*.

Example 2.2 Let $S = \{a, b, c\}$ be a set with its diagonal

$$\triangle_S = \{(a, a), (b, b), (c, c)\}$$

as the equality relation. If we denote by $K = \triangle_S \cup \{(a, b), (b, a)\}$, then we can define the apartness $\neq$ on S to be $(S \times S) \setminus K$. Then $(S, =, \neq)$ is a set with apartness. The relation $\alpha \subseteq S \times S$, defined by

$$\alpha = \{(c, a), (c, b)\},$$

is a coquasiorder on S.

Like the machinery described in [23, 24] for groups and commutative rings with tight apartness, here the machinery of equivalences for a set with apartness is presented in 'dual' terms in analogy with the relation apartness/equality.

For two relations α and β defined on S we say that α is *associated* with β if

$$\forall x, y, z \in S\ ((x, y) \in \alpha \wedge (y, z) \in \beta \Rightarrow (x, z) \in \alpha).$$

A coquasiorder κ defined on S that is symmetric, that is,

$$(x, y) \in \kappa \Rightarrow (y, x) \in \kappa, \qquad \text{(Q3)}$$

is called a *coequivalence* on S.

Theorem 2.3 *If κ is a coequivalence on S, then the relation $\sim\kappa = \{(x, y) : (x, y) \rhd \kappa\}$ is an equivalence on S that κ is associated with $\sim\kappa$.*

P r o o f. As the reflexivity and symmetry are almost obvious, we prove only the transitivity. If $(x, y) \in \sim\kappa$ and $(y, z) \in \sim\kappa$, then, by the definition of $\sim\kappa$, we have that $(x, y) \rhd \kappa$ and $(y, z) \rhd \kappa$. For an element $(a, b) \in \kappa$, by cotransitivity of κ, we have $(a, x) \in \kappa$ or $(x, y) \in \kappa$ or $(y, z) \in \kappa$ or $(z, b) \in \kappa$. Thus $(a, x) \in \kappa$ or $(z, b) \in \kappa$, which implies that $a \neq x$ or $b \neq z$, i.e., $(x, z) \neq (a, b)$. So $(x, z) \rhd \kappa$ and $(x, z) \in \sim\kappa$. Therefore $\sim\kappa$ is an equivalence on S.

Let $(x, y) \in \kappa$ and $(y, z) \in \sim\kappa$; thus $(x, y) \in \kappa$ and $(y, z) \rhd \kappa$. By the symmetry and cotransitivity of κ we have $(x, z) \in \kappa$ or $(y, z) \in \kappa$. Thus $(x, z) \in \kappa$, and κ is associated with $\sim\kappa$. □

Recall that the set

$$S/\sim\kappa = \{a(\sim\kappa) : a \in S\}$$

($a(\sim\kappa)$ is, as usual, $\sim\kappa$-class of an element a, i.e., it is the set of all elements $x \in S$ such that $(a, x) \in \sim\kappa$), with equality defined by

$$a(\sim\kappa) = b(\sim\kappa) \Leftrightarrow (a, b) \in \sim\kappa,$$

is called a *factor set*. We say that $\sim\kappa$ is an *equivalence determined by the coquasiorder* κ. Coquasiorders are the tool for introducing an apartness relation on a factor set.

Theorem 2.4 *Let S be a set with apartness and let* κ *be a coequivalence on S. Then* $(S/\sim\kappa, =, \neq)$ *where*

$$a(\sim\kappa) = b(\sim\kappa) \Leftrightarrow (a, b) \rhd \kappa$$

$$a(\sim\kappa) \neq b(\sim\kappa) \Leftrightarrow (a, b) \in \kappa$$

is a set with apartness. Moreover, the quotient mapping $\pi : S \to S/\sim\kappa$*, defined by* $\pi(x) = x(\sim\kappa)$*, is an onto se-mapping.*

P r o o f. The irreflexivity of $\neq$ is implied by its definition and by the irreflexivity of κ.

Let $a(\sim\kappa) \neq b(\sim\kappa)$; then $(a, b) \in \kappa$ implies that $(b, a) \in \kappa$, that is $b(\sim\kappa) \neq a(\sim\kappa)$.

Let $a(\sim\kappa) \neq b(\sim\kappa)$ and $u(\sim\kappa) \in S/\sim\kappa$. Then $(a, b) \in \kappa$, and, by the cotransitivity of κ, we have $(a, u) \in \kappa$ or $(u, b) \in \kappa$. Finally we have that $a(\sim\kappa) \neq u(\sim\kappa)$ or $u(\sim\kappa) \neq b(\sim\kappa)$, so the relation $\neq$ is cotransitive.

Let $\pi(x) \neq \pi(y)$, i.e., $x(\sim\kappa) \neq y(\sim\kappa)$, which, by what we have just proved, means that $(x, y) \in \kappa$. Then, by the consistency of κ, we have $x \neq y$. So π is an se-mapping.

Let $a(\sim\kappa) \in S/\sim\kappa$ and $x \in a(\sim\kappa)$. Then $(a, x) \in \sim\kappa$, i.e., $a(\sim\kappa) = x(\sim\kappa)$, which implies that $a(\sim\kappa) = x(\sim\kappa) = \pi(x)$. Thus π is an onto mapping. □

The main result of this section is the following.

Theorem 2.5 *Let* $f : S \longrightarrow T$ *be an se-mapping between sets with apartness. Then:*

1. *the relation*

$$\text{coker } f \equiv \{(x, y) \in S \times S : f(x) \neq f(y)\}$$

is a coequivalence on S (which we call the cokernel of f);

2. coker f *is associated with the kernel of f—denoted, as usual, by* ker f*— and*

$$\ker f \subseteq \sim\text{coker } f;$$

3. $(S/\ker f, =, \neq)$ *is a set with apartness, where*

$$a(\ker f) = b(\ker f) \Leftrightarrow (a, b) \in \ker f$$

$$a(\ker f) \neq b(\ker f) \Leftrightarrow (a, b) \in \text{coker } f;$$

4. *the mapping* $\theta : S/\ker f \longrightarrow T$*, defined by* $\theta(x(\ker f)) = f(x)$*, is a one-one, injective se-mapping such that* $f = \theta \circ \pi$*; and*
5. *if f maps S onto T, then* θ *is an apartness bijection.*

Proof. (i) The consistency of coker f is easy to prove: if $(x, y) \in$ coker f, then $f(x) \neq f(y)$ and therefore $x \neq y$.

If $(x, y) \in$ coker f, then, by the symmetry of apartness in T, $f(y) \neq f(x)$; so $(y, x) \in$ coker f.

If $(x, y) \in$ coker f and $z \in S$—i.e., $f(x) \neq f(y)$ and $f(z) \in T$—then either $f(x) \neq f(z)$ or $f(z) \neq f(y)$; that is, either $(x, z) \in$ coker f or $(z, y) \in$ coker f. Hence coker f is a coequivalence on S.

(ii) Let $(x, y) \in$ coker f and $(y, z) \in \ker f$; then $f(x) \neq f(y)$ and $f(y) = f(z)$. Hence $f(x) \neq f(z)$—that is, $(x, z) \in$ coker f—and coker f is associated with $\ker f$.

Now let $(x, y) \in \ker f$, so $f(x) = f(y)$. If $(u, v) \in$ coker f, then, by the cotransitivity of coker f, it follows that $(u, x) \in$ coker f or $(x, y) \in$ coker f or $(y, v) \in$ coker f. Thus either $(u, x) \in$ coker f or $(y, v) \in$ coker f, and, by the consistency of coker f, either $u \neq x$ or $y \neq v$; whence we have $(x, y) \neq (u, v)$. Thus $(x, y) \triangleright$ coker f, or, equivalently $(x, y) \in \sim$coker f.

(iii) This follows from the definition of $\neq$ in $S/\ker f$ and (i).

(iv) Let us first prove that θ is well defined. Let $x(\ker f), y(\ker f) \in S/\ker f$ be such that $x(\ker f) = y(\ker f)$; that is, $(x, y) \in \ker f$. Then we have $f(x) = f(y)$, which, by the definition of θ, means that $\theta(x(\ker f)) = \theta(y(\ker f))$. Now let $\theta(x(\ker f)) = \theta(y(\ker f))$; then $f(x) = f(y)$. Hence $(x, y) \in \ker f$, which implies that $x(\ker f) = y(\ker f)$. Thus θ is one-one.

Next let $\theta(x(\ker f)) \neq \theta(y(\ker f))$; then $f(x) \neq f(y)$. Hence $(x, y) \in$ coker f, which, by (iii), implies that $x(\ker f) \neq y(\ker f)$. Thus θ is an se-mapping.

Let $x(\ker f) \neq y(\ker f)$; that is, by (iii), $(x, y) \in$ coker f. So we have $f(x) \neq f(y)$, which, by the definition of θ means $\theta(x(\ker f)) \neq \theta(y(\ker f))$. Thus θ is injective. On the other hand, by the definition of composition of functions, Theorem 2.4, and the definition of θ, for each $x \in S$ we have

$$(\theta \circ \pi)(x) = \theta(\pi(x)) = \theta(x(\ker f)) = f(x).$$

(v) Taking into account (iv), we have to prove only that θ is onto. Let $y \in T$. Then, as f is onto, there exists $x \in S$ such that $y = f(x)$. On the other hand $\pi(x) = x(\ker f)$. By (iv), we now have

$$y = f(x) = (\theta \circ \pi)(x) = \theta(\pi(x)) = \theta(x(\ker f)).$$

Thus θ is onto. □

Theorem 2.5 is the *Apartness Isomorphism Theorem* for sets with apartness.

3 Semigroups with apartness

We define a notion of a semigroup in a constructive way. In doing so, we follow [12, 14, 23, 24].

A *semigroup with apartness* is a set S with apartness, together with an associative, strongly extensional binary operation, denoted by juxtaposition, on S. For example, if A is a set with apartness, then it is routine to verify that A^A, taken with the standard equality and apartness (as defined in the previous section) and with composition of functions as the binary operation, is a semigroup with apartness.

Until the end of this paper, we adopt the convention that *semigroup* means *semigroup with apartness.*

It is well known that if the standard apartness on the additive semigroup $\mathbb{R}$ is tight, then we can prove the constructively questionable *Markov's principle*:

For each binary sequence $(a_n)_{n\geq 1}$, if it is impossible that $a_n = 0$ for all n, then there exists n with $a_n = 1$. (**MP**)

The following examples show that we cannot prove constructively that the apartness on every *finite* semigroup is tight.

Example 3.1 Let $A = \{0, 1, 2\}$ with the usual equality relation—that is, the diagonal Δ_A of $A \times A$. Let $K = \Delta_A \cup \{(1, 2), (2, 1)\}$, and define an apartness $\neq$ on A by $x \neq y \Leftrightarrow (x, y) \notin K$. Then, as we observed above, $S \equiv A^A$ becomes a semigroup with apartness in a standard way. Define mappings $f, g : A \to A$ by

$$f(0) = 1, \; f(1) = 1, \; f(2) = 2,$$

$$g(0) = 2,\ g(1) = 1, g(2) = 2.$$

In view of our definition of the apartness on A, there is no element x of A with $f(x) \neq g(x)$; so, in particular, f and g are se-functions. However, if $f = g$, then $1 = 2$, which, by our definition of the equality on A, is not the case. Hence the apartness on S is not tight.

Let α be a relation on a semigroup with apartness S. Then α is *compatible* with the semigroup multiplication if

$$\forall a, b, x, y \in S\,((a, b) \in \alpha \wedge (x, y) \in \alpha \ \Rightarrow\ (ax, by) \in \alpha).$$

Let us remember that in **CLASS** the compatibility property is an important condition for providing the semigroup structure on quotient sets. Now we are looking for the tools for introducing apartness relation on a factor semigroup. Our starting point is the results from the previous section, as well as the next definition. The relation α is *cocompatible* with the semigroup multiplication if

$$\forall a, b, x, y \in S\,((ax, by) \in \alpha \ \Rightarrow\ (a, b) \in \alpha \vee (x, y) \in \alpha).$$

By a *cocongruence* on a semigroup S we mean a coequivalence on S that is cocompatible with multiplication.

Theorem 3.2 *If κ is a cocongruence on a semigroup S, then the relation $\sim\kappa = \{(x, y) : (x, y) \triangleright \kappa\}$ is a congruence on S such that κ is associated with $\sim\kappa$.*

P r o o f. By Theorem 2.3, $\sim\kappa$ is an equivalence on S such that κ is associated with $\sim\kappa$. If $(a, b), (x, y) \in \sim\kappa$ then for any $(u, v) \in \kappa$ we have both $(a, b) \neq (u, v)$ and $(x, y) \neq (u, v)$. Now, we also have $(u, ax) \in \kappa$ or $(ax, by) \in \kappa$ or $(by, v) \in \kappa$. If $(ax, by) \in \kappa$, then by the cocompatibility of κ, either $(a, b) \in \kappa$ or $(x, y) \in \kappa$, which is impossible. Thus $(u, ax) \in \kappa$ or $(by, v) \in \kappa$; so either $u \neq ax$ or $by \neq v$, and therefore $(ax, by) \neq (u, v)$. Hence $(ax, by) \triangleright \kappa$. Thus $(ax, by) \in \sim\kappa$, and $\sim\kappa$ is a congruence on S. □

Let S and T be semigroups with apartness. A mapping $f : S \to T$ is a *homomorphism* if

$$\forall x, y \in S\,(f(xy) = f(x)f(y)).$$

An se-homomorphism is an *apartness embedding* if it is one-one and injective. An se-homomorphism is an *apartness isomorphism* if it is an apartness bijection.

Theorem 3.3 *Let S be a semigroup with apartness, and let κ be a cocongruence on S. Define*

$$a(\sim\kappa) = b(\sim\kappa) \ \Leftrightarrow\ (a, b) \triangleright \kappa,$$
$$a(\sim\kappa) \neq b(\sim\kappa) \ \Leftrightarrow\ (a, b) \in \kappa,$$
$$a(\sim\kappa)\, b(\sim\kappa) = (ab)(\sim\kappa).$$

Then $(S/\sim\kappa, =, \neq, \cdot)$ is a semigroup with apartness. Moreover, the quotient mapping $\pi : S \to S/\sim\kappa$, defined by $\pi(x) = x(\sim\kappa)$, is a se-homomorphism of S onto $S/\sim\kappa$.

P r o o f. By Theorem 2.4, $(S/\sim\kappa, =, \neq)$ is a set with apartness. The associativity of multiplication in $S/\sim\kappa$ follows from that of multiplication on S.

Let $a(\sim\kappa)x(\sim\kappa) \neq b(\sim\kappa)y(\sim\kappa)$; then $(ax)(\sim\kappa) \neq (by)(\sim\kappa)$. By Theorem 2.4, we have that $(ax, by) \in \kappa$. But κ is a cocongruence, so either $(a, b) \in \kappa$ or $(x, y) \in \kappa$. Thus, by the definition of $\neq$ in $S/\sim\kappa$, either $a(\sim\kappa) \neq b(\sim\kappa)$ or $x(\sim\kappa) \neq y(\sim\kappa)$. So $(S/\sim\kappa, =, \neq, \cdot)$ is a semigroup with apartness. Using that fact and the definition of π, we have

$$\pi(xy) = (xy)(\sim\kappa) = x(\sim\kappa)\, y(\sim\kappa) = \pi(x)\pi(y).$$

Hence π is a homomorphism, and, by Theorem 2.4, π is an se-mapping onto $S/\sim\kappa$. □

The main result of this section, as well as the whole paper, is the following

Theorem 3.4 *Let $f : S \longrightarrow T$ be an se-homomorphism between semigroups with apartness. Then:*

1. *the relation* coker f *is a cocongruence on S associated with* ker f*;*

2. $(S/\ker f, =, \neq, \cdot)$ *is a semigroup with apartness, where*

$$a(\ker f) = b(\ker f) \Leftrightarrow (a, b) \in \ker f,$$
$$a(\ker f) \neq b(\ker f) \Leftrightarrow (a, b) \in \operatorname{coker} f,$$
$$a(\ker f)\, b(\ker f) = (ab)(\ker f);$$

3. *the mapping* $\theta : S/\ker f \longrightarrow T$, *defined by* $\theta(x(\ker f)) = f(x)$, *is an se-embedding such that* $f = \theta \circ \pi$; *and*
4. *if* f *is onto, then* θ *is an apartness isomorphism.*

P r o o f. (i) Taking into account Theorem 2.5, it is enough to prove that coker f is cocompatible with multiplication in S. Let $(ax, by) \in \operatorname{coker} f$—i.e., $f(ax) \neq f(by)$. Since f is a homomorphism, we have $f(a)f(x) \neq f(b)f(y)$. The strong extensionality of multiplication implies that either $f(a) \neq f(b)$ or $f(x) \neq f(y)$. Thus either $(a, b) \in \operatorname{coker} f$ or $(x, y) \in \operatorname{coker} f$, and therefore coker f is a cocongruence on S.

(ii) This follows by Theorem 2.5 and Theorem 3.3.

(iii) Using (ii) and the assumption that f is a homomorphism, we have

$$\begin{aligned}\theta(x(\ker f)\, y(\ker f)) &= \theta((xy)(\ker f))\\ &= f(xy)\\ &= f(x)f(y)\\ &= \theta(x(\ker f))\,\theta(y(\ker f)).\end{aligned}$$

By Theorem 2.5, θ is a one-one, injective se-homomorphism—that is, an se-embedding.

(iv) This follows by Theorem 2.5 and (iii). □

Theorem 3.4 is the *Apartness Isomorphism Theorem* for semigroups with apartness. As we have already pointed out above, coequivalences and cocongruences are important tools for introducing an apartness relation on a factor set and a factor semigroup. We use the prefix "co" in these notions to emphasise their "cooperation" with their classical "friends", equivalences and congruences. The results of such a cooperation are the *apartness isomorphism theorems* presented in Sections 2 and 3.

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